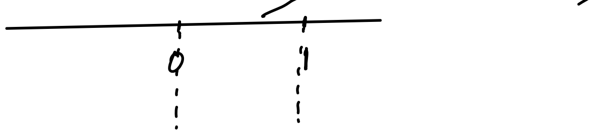
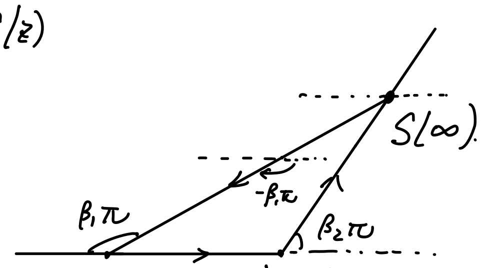
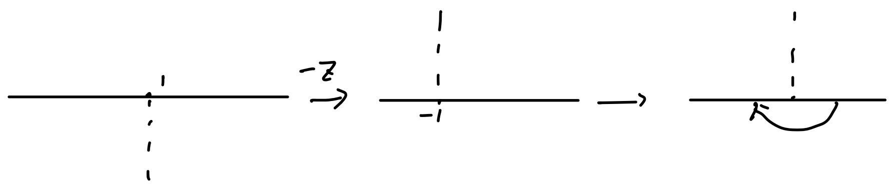
21. $\displaystyle\int_0^z z^{-\beta_1}(1-z)^{-\beta_2}\,dz = S(z)$





$k = \int_0^1 x^{-\beta_1}(1-x)^{-\beta_2}dx$

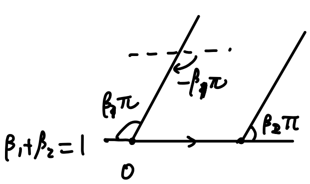
$$z^{-\beta_1}(1-z)^{-\beta_2} = \begin{cases} x^{-\beta_1}(1-x)^{-\beta_2}, & 0<x<1 \\ |x|^{-\beta_1}\cdot e^{-\beta_1 \cdot i\pi}(1-x)^{-\beta_2} & x<0 \\ x^{-\beta_1}(x-1)^{-\beta_2}\cdot e^{+\beta_2(+i\pi)} & x>1 \end{cases}$$

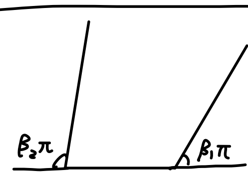


$x>1,\ \displaystyle\int_0^{z} = \underbrace{\int_0^1}_{k} + \left(\int_1^{z} x^{-\beta_1}(x-1)^{-\beta_2}dx\right)\cdot e^{i\beta_2\pi}$

$L_1 = \int_1^\infty x^{-\beta_1}(x-1)^{-\beta_2}dx$ $\qquad \beta_1+\beta_2>1$

$L_2 = \int_{-\infty}^1 |x|^{-\beta_1}(1-x)^{-\beta_2}dx$





Entire function

$(-z_1)(1-\frac{z}{z_1})$

- polynomial $\underline{P(z) = a_0 \cdot (z-z_1) \cdots (z-z_n)} = b_0 (1-\frac{z}{z_1}) \cdots (1-\frac{z}{z_n})$
- function f with no zero on $\mathbb{C} \iff f(z) = e^{g(z)}$. g hol.
- $\sin(z)$: $0, \pm\pi, \pm 2\pi, \cdots$

$\sin(z) = -\sin(z+\pi)$

$$\frac{\sin(z)}{\pi} = z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

If f entire with infinitely many zeros: 0, $a_1, \cdots, a_n, \cdots$

$f \not\equiv 0$

then $\underline{|a_n| \xrightarrow{n \to +\infty} +\infty}$.

$\forall R, \exists N$ s.t. $|a_n| > R$ if $n \geq N$.

$\Updownarrow$

$\underline{\exists R, \text{ s.t. } \forall N \ \exists n > N \text{ s.t. } |a_n| \leq R}$

$$\underline{h = z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)} \sim f \qquad \frac{f}{h} = e^g \Rightarrow f = h \cdot e^g$$

$\uparrow$ may not converge! ($\frac{z}{a_n} \to 0$)

$$\prod_{n=1}^{\infty} (1+a_n) = \lim_{N \to +\infty} \prod_{n=1}^{N} (1+a_n) \text{ converges} \Rightarrow a_n \to 0$$

$\prod_{n=1}^{N} (1+a_n) = P_N$

$$\frac{P_{N+1}}{P_N} = 1 + a_n \longrightarrow \frac{P}{P} = 1 \Rightarrow a_n \to 0.$$

$$\boxed{\prod_{n=1}^{N} (1+a_n) = e^{\sum_{n=1}^{N} \underline{\operatorname{Log}(1+a_n)}}}$$

$(1-\varepsilon)|a_n| \leq \boxed{|\operatorname{Log}(1+a_n)| \leq 2|a_n| \text{ when } |a_n| \leq \frac{1}{2}}$

$$|\operatorname{Log}(1+a_n)| = \left|a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots\right| \leq |a_n| + \frac{|a_n|^2}{2} + \frac{|a_n|^3}{3} + \cdots = |\log(1-|a_n|)| \leq 2|a_n|.$$

$a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots = a_n \cdot \left(1 - \frac{a_n}{2} + \frac{a_n^2}{3} - \cdots\right)$

$|\log(1-x)| \leq 2x$ when $x < \frac{1}{2}$

$$\frac{d}{dx}\Big(\underbrace{-\log(1-x)-2x}_{f(x)}\Big)=+\frac{+1}{1-x}-2\le 0 \qquad 1\le$$

$$\boxed{\sum_n |a_n| \text{ convergent} \Leftrightarrow \sum_{n=1}^{N} \operatorname{Log}(1+a_n) \text{ abs. conv.} \Rightarrow \prod_{n=1}^{N}(1+a_n) \text{ convergent.}}$$

$$\left(f(0)=0. \Rightarrow -\log(1-x)-2x\le 0 \quad \Downarrow \quad x<\tfrac{1}{2} \quad 1-x>\tfrac{1}{2} \quad \tfrac{1}{1-x}<2\right)$$

Prop: $\{F_n\}$ seq. hol. fcts. on Ω. $|F_n(z)-1|\le C_n$

$$\sum_n C_n \text{ converges} . \Rightarrow \prod_{n=1}^{\infty} F_n(z) \text{ converges uniformly to } F(z).$$

$$F_n(z) = 1+\underbrace{(F_n(z)-1)}_{a_n(z)}$$

$\{a_n\}\subset\mathbb{C}$ seq. $|a_n|\xrightarrow{n\to+\infty}+\infty$

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{P_n(z)} = e^{\sum_{n=1}^{\infty}\left(\operatorname{Log}\left(1-\frac{z}{a_n}\right)+P_n(z)\right)}$$

Want $\displaystyle\sum_{n=1}^{\infty}\Big(\underbrace{\operatorname{Log}\left(1-\frac{z}{a_n}\right)+P_n(z)}\Big)$ $r_n(z)$ absolutely converge.

$\displaystyle\Leftrightarrow \sum_{n=1}^{\infty}\left(\left(1-\frac{z}{a_n}\right)e^{P_n(z)}-1.\right)$ absolutely converge

$$\left(\sum_{n=1}^{\infty}|a_n| \Leftrightarrow \sum_{n=1}^{\infty}\log(1+a_n) \text{ absolutely converge}\right)$$

$$\left|\operatorname{Log}\left(1-\frac{z}{a_n}\right)\right| = \left|-\frac{z}{a_n}-\frac{1}{2}\left(\frac{z}{a_n}\right)^2-\frac{1}{3}\left(\frac{z}{a_n}\right)^3-\cdots\right| \le \sum_n \frac{|z|}{|a_n|}\left(1-\frac{R}{|a_n|}\right)^{-1} \qquad |a_n|\sim n^{\frac{1}{p}}$$

$$\sum\frac{1}{n^p}$$

$$P_n(z) = \frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\frac{1}{3}\left(\frac{z}{a_n}\right)^3+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}$$

$$\boxed{\sum\frac{1}{|a_n|} \text{ conveg}}$$

$$\Rightarrow \underbrace{\mathrm{Log}\left(1-\frac{z}{a_n}\right)+P_n(z)}_{r_n(z)} = -\frac{1}{m_n+1}\left(\frac{z}{a_n}\right)^{m_n+1} - \cdots$$

$|z| \le R$

$\sum \frac{1}{|a_n|} \leadsto \sum \frac{1}{|a_n|^{m_n+1}}$

$$\boxed{|r_n(z)| \le \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\cdot\left(1-\frac{R}{|a_n|}\right)^{-1}}$$

$m_n = 0$: $\le C$

$m_n = 1$: $\frac{1}{2}\left(\frac{R}{|a_n|}\right)^2$

$\sum \frac{1}{2}\left(\frac{R}{|a_n|}\right)^2$

$\frac{R}{|a_n|} \to 0$

$\frac{1}{|a_n|^2} \to 0$

$|a_n| \sim n^{\frac{1}{4}}$

$m_n = n$

$$\boxed{\sum_n |r_n(z)| \le \sum_n \frac{1}{n+1}\left(\frac{R}{|a_n|}\right)^{n+1}\cdot C}$$

$\frac{R}{|a_n|} \le \frac{1}{2} < 1$

$0 \notin \{a_n\} \to \infty$

$$h(z) = z^m \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\cdot e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}$$

$m_n = n$

$$\boxed{= z^m \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{n}\left(\frac{z}{a_n}\right)^{n}} = h(z)}$$

zeros of h: 0 of order m and $\{a_n\}$

$$\boxed{f = h\cdot e^{g(z)}}$$

Thm (Weierstrass) Given any seq. $\{a_n\}$, $|a_n| \to +\infty$. there exists an entire fct. f that vanishes at all $z = a_n$ and nowhere else. Any other such entire fct. is of the norm $f(z)\cdot e^{g(z)}$.

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{m_n+1}} \frac{1}{m_n+1}$$

Q: Is there some $k>1$ s.t.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k}}}$$

$\{a_n\} \to +\infty$

$a_n \sim \log n$

$$\boxed{\sum_n \frac{1}{(\log n)^p} > \sum \frac{1}{n}}$$

Hadamard's factorization

Thm: Suppose f is entire and has growth order ρ_0. Let k be the integer s.t. $k \le \rho_0 < k+1$. If $a_1, a_2, \dots$ denote (non-zero) zeros of f. Then

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

$P(z)$: polynomial of deg $\le k$

$$E_k\left(\frac{z}{a_n}\right) = \left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\dots+\frac{1}{k}\left(\frac{z}{a_n}\right)^k}$$

Def: Let f be an entire fct. If there exist $\rho>0$, $A, B>0$ s.t.

$$|f(z)| \le A \cdot e^{B|z|^\rho} \text{ for all } z \in \mathbb{C}$$

we say that f has an order of growth $\le \rho$.

$$\boxed{\text{Order of growth of } f = \rho_f = \inf \rho}$$

$\left(\forall \varepsilon>0, \exists A, B>0 \text{ s.t. } |f(z)| \le A \cdot e^{B|z|^{\rho+\varepsilon}} \text{ for } z \in \mathbb{C}.\right)$

$\rightsquigarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{|a_n|^s} \text{ converges}}$ for any $s > \rho \geq \rho_f$

$$\boxed{\text{Ex: } \quad \frac{\sin z}{\pi} = z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).}$$