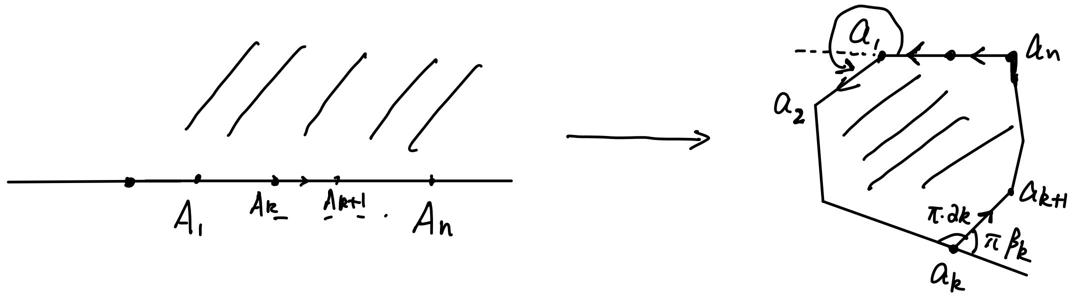


$$S(z)=\int_0^z \frac{d\zeta}{(\zeta-A_1)^{\beta_1}\cdots(\zeta-A_n)^{\beta_n}}$$

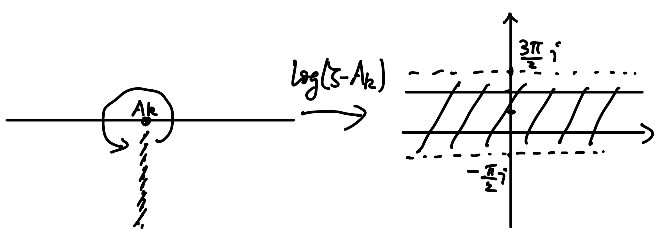
$\sum_i \beta_i = 2$

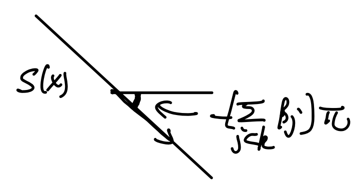
$0<\beta_i<1.$

$$(\zeta-A_k)^{\beta_k} = \begin{cases} (x-A_k)^{\beta_k} & \zeta=x>A_k \\ (A_k-x)^{\beta_k}\cdot e^{i\beta_k\pi}, & \zeta=x<A_k \end{cases}$$

$\shortparallel$

$e^{\beta_k\cdot\log(\zeta-A_k)}$





$$S(x)=\int_0^x \frac{dx}{\prod_{j\le k}(x-A_j)^{\beta_j}\ \prod_{j>k}(x-A_j)^{\beta_j}} \ : \ [A_k,A_{k+1}] \to \mathbb{C}$$

ψ

$$S'(x)=\frac{1}{\prod_{j<k}(x-A_j)^{\beta_j}\ \prod_{j>k}(x-A_j)^{\beta_j}}$$

$\shortparallel$ $\qquad$ $\shortparallel$

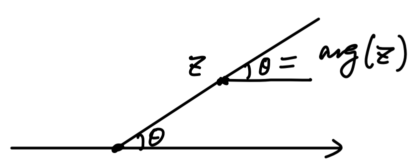
$\prod_{j<k}(x-A_j)^{\beta_j}$ $\qquad$ $\prod_{j>k}(A_j-x)^{\beta_j}e^{i\beta_j\cdot\pi}$

$$\boxed{\arg S'(x) = -\Big(\sum_{j>k}\beta_j\Big)\pi}$$

$\shortparallel$

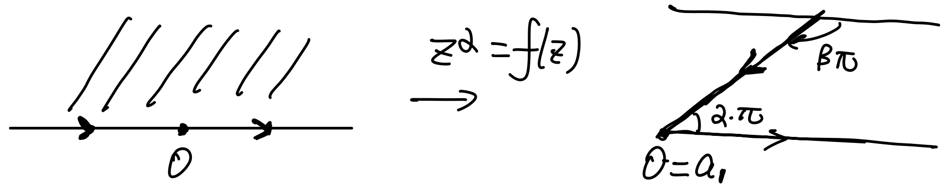
$R(x)\cdot e^{-i(\sum_{j>k}\beta_j)\pi}$

$\cdots<A_{k-1}<A_k<x<A_{k+1}<A_{k+2}<\cdots$



$0 < \alpha < 2.$

Ex:



$\beta = 1-\alpha.$

$$z^\alpha = \int_0^z f'(\zeta)\,d\zeta = \alpha\int_0^z \zeta^{\alpha-1}\,d\zeta = \alpha\cdot\int_0^z \frac{1}{\zeta^\beta}\,d\zeta$$

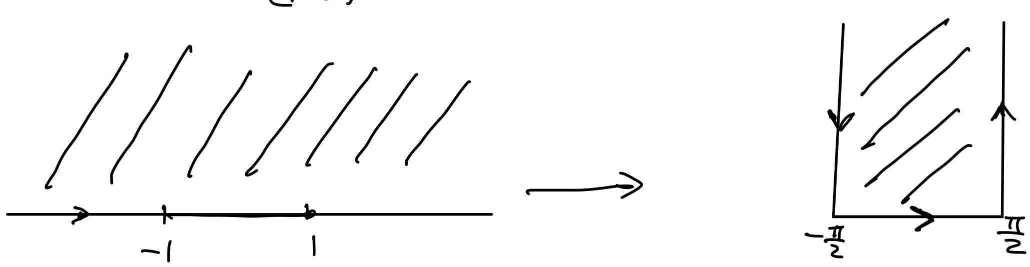
$$S(z) = \alpha\int_0^z \frac{1}{\zeta^\beta}\,d\zeta. \qquad \zeta^\beta = \begin{cases} x^\beta & \zeta = x > 0 \\ |x|^\beta\cdot e^{i\beta\pi} & \zeta = x < 0 \end{cases}$$

$$S'(x) = \begin{cases} \alpha\cdot|x|^{-\beta}\cdot e^{-i\beta\pi} & x < 0 \\ \alpha\cdot x^{-\beta} & x > A_1 \end{cases}$$

Ex: $f(z) = \int_0^z \dfrac{d\zeta}{(1-\zeta^2)^{1/2}}$. $\quad (1-\zeta^2)^{1/2} = (1-\zeta)^{1/2}\cdot(1+\zeta)^{1/2}$

$f(x) = \int_0^x \dfrac{d\zeta}{(1-\zeta^2)^{1/2}} = \arcsin(x).$

$[-1,1] \longmapsto [-\frac{\pi}{2}, \frac{\pi}{2}].$

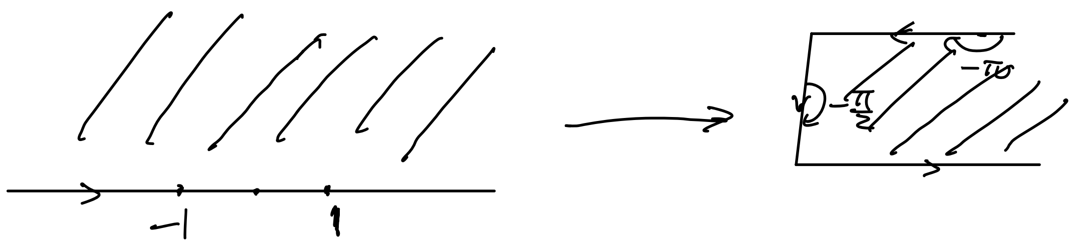


$x < -1.$ $\quad (1-\zeta)^{1/2}(1+\zeta)^{1/2} = (1-x)^{1/2}\cdot|x+1|\cdot e^{\frac{1}{2}\cdot\pi i}$. $\quad \arg f'(x) = -\frac{\pi}{2}$

$-1 < x < 1.$ $\quad (1-x^2)^{1/2} > 0 \qquad \arg f'(x) = 0$

$x > 1$

$$(\zeta-1)^{\frac{1}{2}}(\zeta+1)^{1/2} = (x-1)^{1/2}(x+1)^{1/2} \qquad \arg = 0$$



$$S(z)=\int_0^z \frac{d\zeta}{(\zeta+1)^{1/2}(\zeta-1)^{1/2}} = (-i)\, f(z).$$

$\zeta = x < -1.$ $\quad |\zeta+1|^{1/2}\cdot e^{i\pi\cdot\frac{1}{2}} \cdot |\zeta-1|\cdot e^{i\cdot\pi\cdot\frac{1}{2}}$ $\quad \arg S'(x) = -\pi$

$-1 < \zeta = x < 1.$ $\quad |\zeta+1|^{1/2}\cdot |\zeta-1|^{1/2} e^{i\cdot\pi\cdot\frac{1}{2}}$ $\quad \arg S'(x) = -\frac{\pi}{2}$

$\zeta = x > 1.$ $\quad |\zeta+1|^{1/2}\cdot |\zeta-1|^{1/2}$ $\quad \arg S'(x) = 0.$

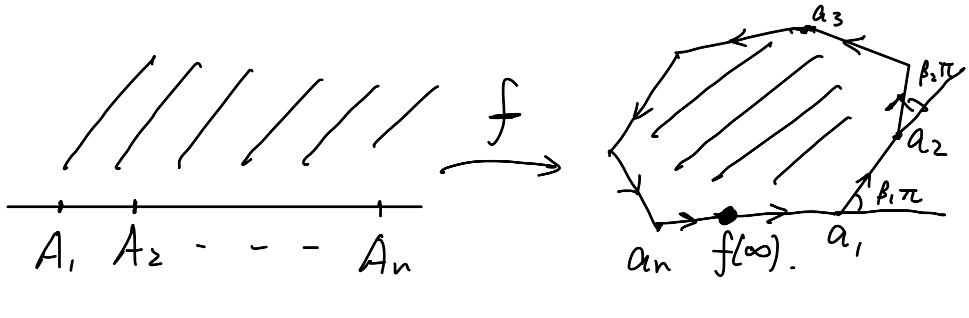
$$S(z)=\int_0^z \frac{d\zeta}{(\zeta-A_1)^{\beta_1}\cdots(\zeta-A_n)^{\beta_n}} \qquad A_1<\cdots<A_n$$

$\zeta = x < A_1,$ $\quad |A_1-\zeta|\, e^{i\beta_1\pi} \qquad |A_n-\zeta|\cdot e^{i\cdot\beta_n\cdot\pi}$

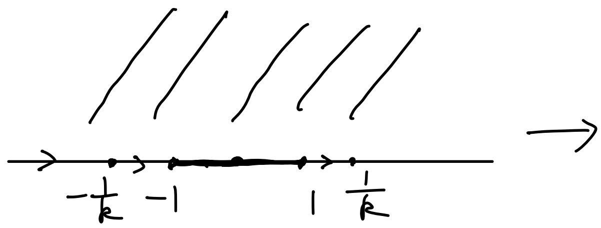
$$\arg S'(x) = -\pi\sum\beta_i = -2\pi.$$

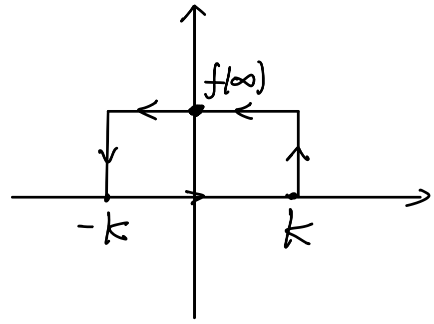
$\longrightarrow$

$A_1 < x < A_2.$ $\quad \arg S'(x) = -\sum_{i>1}\beta_i\pi = -2\pi + \beta_1\pi$



Ex:





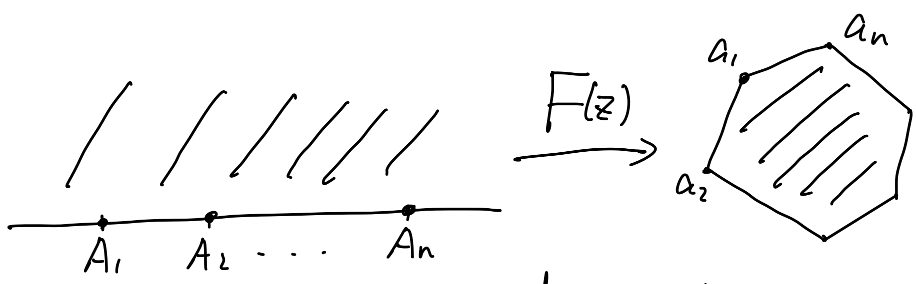
elliptic integral

$$f(z) = \int_0^z \frac{d\zeta}{\left[(1-\zeta^2)\cdot(1-k^2\zeta^2)\right]^{1/2}}$$

$$(1-\zeta^2)\cdot(1-k^2\zeta^2) = (\zeta^2-1)\cdot(k^2\zeta^2-1)$$

$$= \int_0^z \frac{d\zeta}{\left((\zeta^2-1)\cdot(\zeta^2-\frac{1}{k^2})\right)^{1/2}\cdot k}$$

$$= \frac{1}{k}\cdot\int_0^z \frac{d\zeta}{\left[(\zeta+\frac{1}{k})\cdot(\zeta+1)\cdot(\zeta-1)\cdot(\zeta-\frac{1}{k})\right]^{1/2}}$$

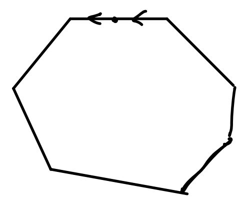


Thm: $\exists$ complex numbers C_1 and C_2 s.t.

$$\boxed{F(z) = C_1\cdot S(z) + C_2}$$

where

$$\boxed{S(z) = \int_0^z \frac{d\zeta}{(\zeta-A_1)^{\beta_1}\cdots(\zeta-A_n)^{\beta_n}}}.$$

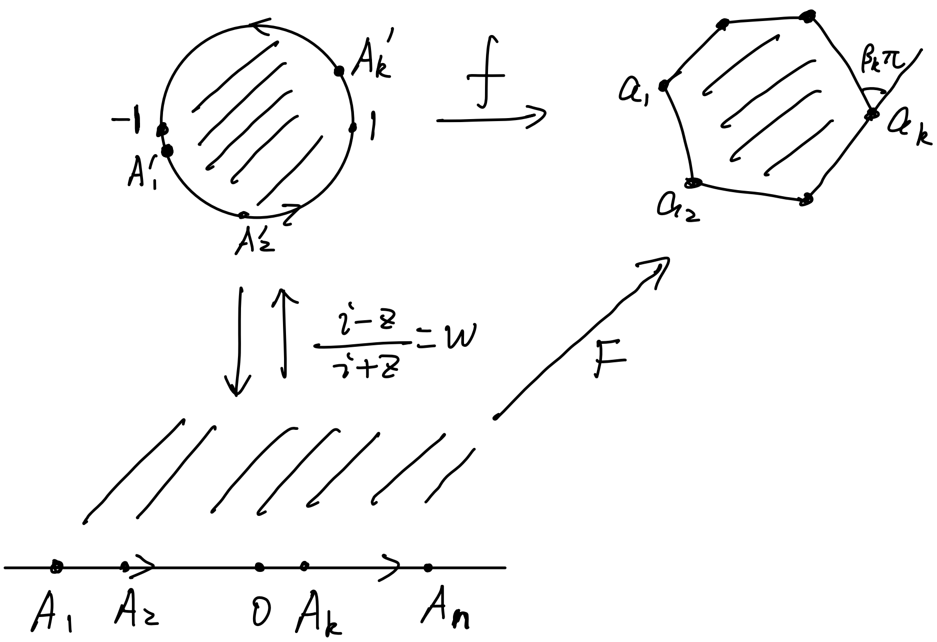


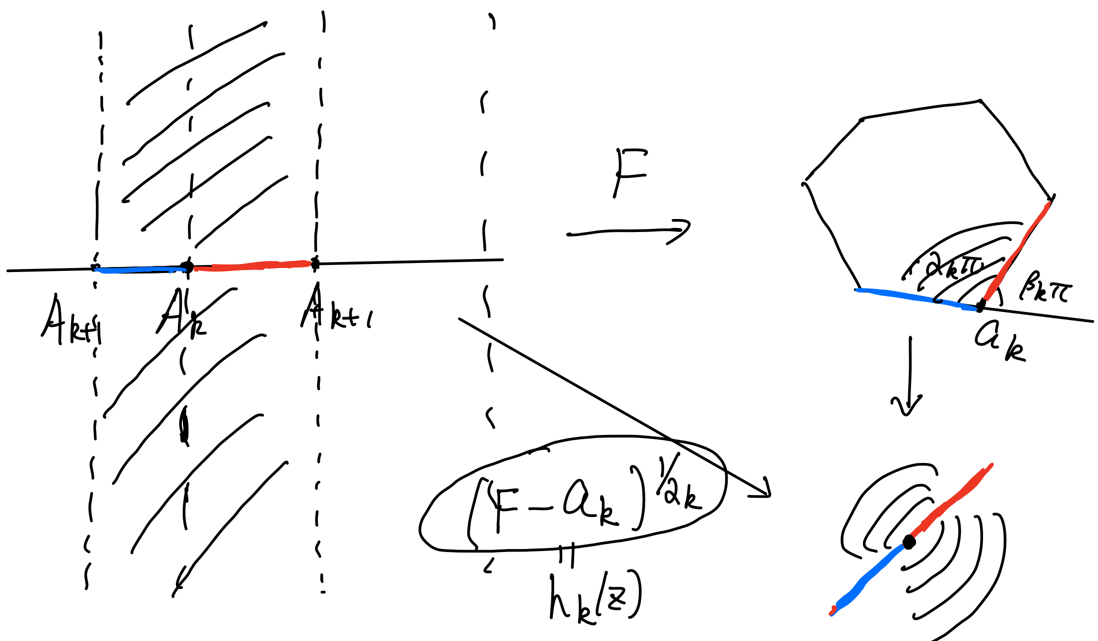
$$F' = C_1\cdot S' = C_1\cdot\frac{1}{(z-A_1)^{\beta_1}\cdots(z-A_n)^{\beta_n}}$$

for all $z\in\mathbb{C}$

$$(\log F')' = \frac{F''}{F'} = \frac{S''}{S'} = \boxed{-\left(\frac{\beta_1}{z-A_1}+\frac{\beta_2}{z-A_2}+\cdots+\frac{\beta_n}{z-A_n}\right) = \frac{F''}{F'}}$$

Thm: Let $f: \mathbb{D} \to P$ conformal mapping. Then f extends to a continuous bijection from $\overline{\mathbb{D}}$ to $\overline{P}$.





$$\frac{F''}{F'} = -\frac{\beta_k}{z-A_k} + \underbrace{E_k(z)}_{\text{holomorphic on } A_{k-1}<\mathrm{Re}(z)<A_{k+1}}$$