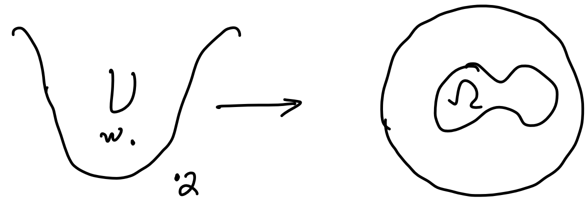
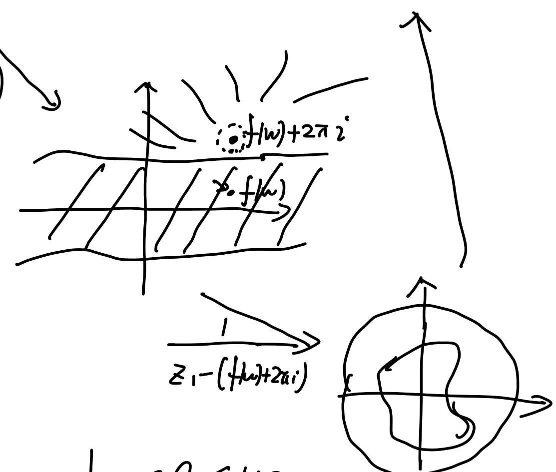
Thm (Riemann Mapping Thm) If $U \subsetneq \mathbb{C}$ simply connected domain. Then $\exists\ f: U \to \mathbb{D}$ holomorphic and bijective.

<u>Proof</u>: <u>Step 1</u>:



$e^{f(z)} = z-a \Leftarrow f(z) = \log(z-a)$



suppose $\underset{\substack{\cup \\ z_n}}{U}$ s.t. $\underline{f(z_n) \longrightarrow f(w) + 2\pi i}$

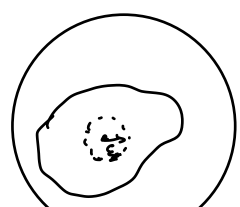
$\Rightarrow \underset{\substack{\shortparallel \\ z_n - a}}{e^{f(z_n)}} \longrightarrow e^{f(w)+2\pi i} = \underset{\substack{\shortparallel \\ w-a}}{e^{f(w)}}$

$\Rightarrow z_n \to w \Rightarrow f(z_n) \to f(w)$.

$\Rightarrow \quad g(z) = \dfrac{1}{f(z) - (f(w)+2\pi i)} \qquad |g(z)| \le \dfrac{1}{\delta} \le C < +\infty$

$\Rightarrow$ rescale $\quad \lambda \cdot g(z) : U \longmapsto \overset{\substack{0 \\ \cap}}{\Omega} \subset \mathbb{D}$. $\qquad \varphi_a(z) = \dfrac{a-z}{1-\bar{a}z}$

- $\mathcal{F} = \{ \underline{f: \Omega \to \mathbb{D}},\ f \text{ injective},\ f(0) = 0 \}$



Cauchy's inequality: $|f'(0)| \le \dfrac{1}{2\pi} \displaystyle\int_{C_\varepsilon} \frac{|f(\zeta)|}{\varepsilon^2} |d\zeta|$

$\shortparallel$

$\dfrac{1}{2\pi} \cdot \dfrac{1}{\varepsilon^2} \cdot 2\pi\varepsilon = \dfrac{1}{\varepsilon} < +\infty$.

$\Rightarrow S = \sup\limits_{f \in \mathcal{F}} |f'(0)| < +\infty$. $\quad \boxed{\exists\ f_n \in \mathcal{F} \text{ s.t. } |f_n'(0)| \to S}$

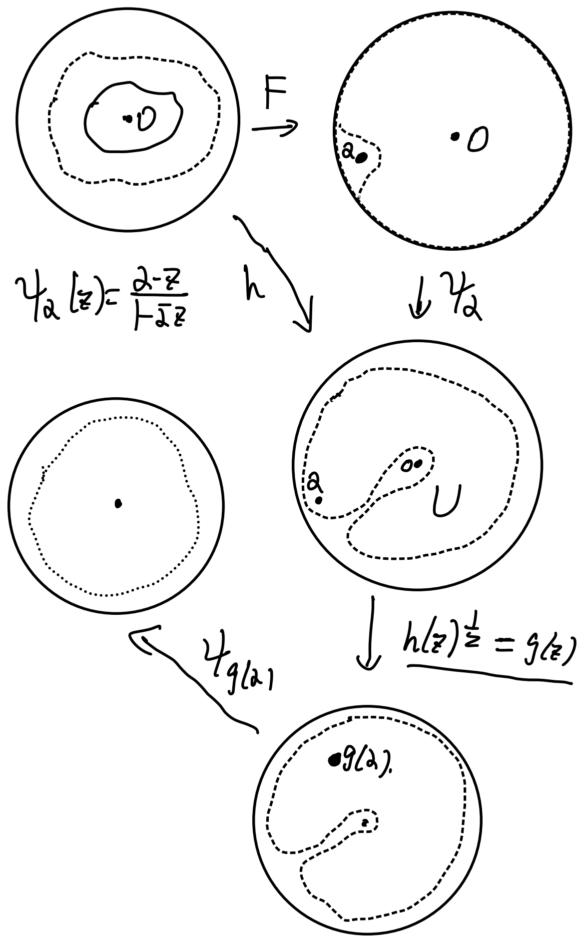
- By Montel's Thm, $\exists \{f_{n_k}\} \subset \mathcal{F}$ s.t. $f_{n_k}(z) \to F(z)$ locally uniformly (uniformly on cpt sets).

$\Rightarrow$ · $F(z)$ is injective. $f(0)=0$

· $|F'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|$

- Prove: F is surjective.

suppose $F(z) \neq \alpha$, $\alpha \in \mathbb{D}$



$h(z) = \varphi_\alpha \circ F : \Omega \to h(\Omega) \not\ni 0$

$g(z) = h(z)^{\frac{1}{2}} = e^{\frac{1}{2}(\log h(z))}$

$$\left(\begin{array}{l} G(z) = \varphi_{g(\alpha)} \circ w^{\frac{1}{2}} \circ \varphi_\alpha \circ F \in \mathcal{F} \\ \Leftrightarrow \underbrace{\varphi_\alpha \circ w^2 \circ \varphi_{g(\alpha)}}_{\Phi} \circ G = F \end{array} \right)$$

$F'(0) = \Phi'(\underset{0}{G(0)}) G'(0)$.

$\Phi = \varphi_\alpha \circ w^2 \circ \varphi_{g(\alpha)} : \mathbb{D} \to \mathbb{D}$.

$\overset{\text{Schwarz}}{\Rightarrow} |\Phi'(0)| \leq 1 \Rightarrow |\Phi'(0)| < 1 \Rightarrow |F'(0)| < |G'(0)|$.

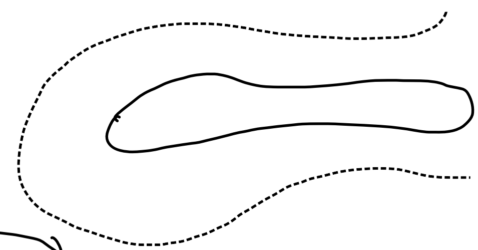
w^2 not injective

$\rightarrow\leftarrow$.

$\left((z^{\frac{1}{2}})' = \frac{1}{2} z^{-\frac{1}{2}} \right)$

$\varphi_\alpha \circ \varphi_\alpha = Id$, $\varphi_\alpha^{-1} = \varphi_\alpha$.

Thm: $\mathcal{F}$ is a family of holomorphic fcts. on Ω that is uniformly bounded on compact subsets of Ω
(Montel)



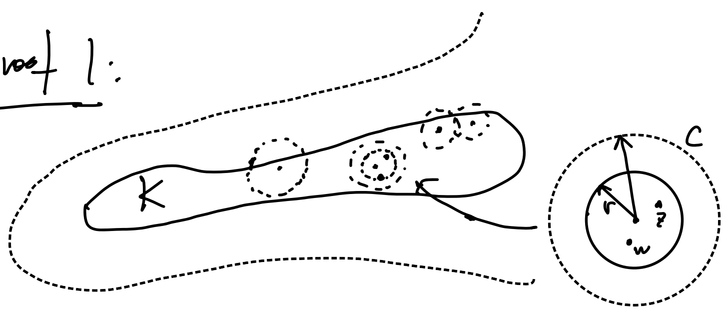
Then every sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence that converges uniformly on every compact subset (to a holomorphic fct. f) $:=$ normal family

Def: $\mathcal{F}$ is equicontinuous on a cpt. set K, if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $z, w \in K$, $|z-w| < \delta$ then $|f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{F}$.

Proof of Montel: 1. $\mathcal{F}$ is equicontinuous on any cpt. subset of Ω.

2. Arzela-Ascoli Thm: $\mathcal{F}$ Equicontinuous on any cpt. subset of Ω $\Rightarrow$ normal family

Proof 1:



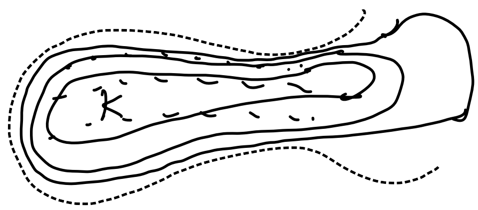
$$\boxed{f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta}$$

$$|f(z) - f(w)| = \left| \frac{1}{2\pi i} \int_{C_{2r}} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) d\zeta \right| \le \frac{1}{\cancel{2\pi}} C_K \cdot \frac{|z-w|}{r^2} \cdot \cancel{2\pi} \cdot 2r$$

$$\left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) = \int_{C_{2r}} f(\zeta) \cdot \frac{(z-w)}{(\zeta - z)(\zeta - w)} d\zeta$$

$$\frac{1}{\cancel{2\pi}} C_K \cdot \frac{|z-w|}{r^2} \cdot \cancel{2\pi} \cdot 2r \le \frac{2C_K}{r} \cdot |z-w|.$$

2. Arzela-Ascoli $\{z_n\}$ sequence of pts. that is dense in Ω.



$\{f_n\} \to \{f_{n,1}\}$ s.t. $f_{n,1}(z_1)$ converges

$\to \{f_{n,2}\}$ s.t. $f_{n,2}(z_2)$ converge

$\to \cdots$

$g_n = \{f_{n,n}\}$ $g_n(z_j)$ converges for any $j \geq 1$.

$$|f_{n,n}(z) - f_{m,m}(z)| = \underbrace{|f_{n,n}(z) - f_{n,n}(z_j)|}_{\underset{\varepsilon}{\wedge}} + |f_{n,n}(z_j) - f_{m,m}(z_j)| < \varepsilon$$

$$+ \underbrace{|f_{m,m}(z_j) - f_{m,m}(z)|}_{\underset{\varepsilon}{\wedge}}$$

$|z - z_j| < \delta \Rightarrow$

$\Rightarrow \{g_n(z)\}$ converges uniformly on K

$K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$ $\Omega = \bigcup_n K_n$

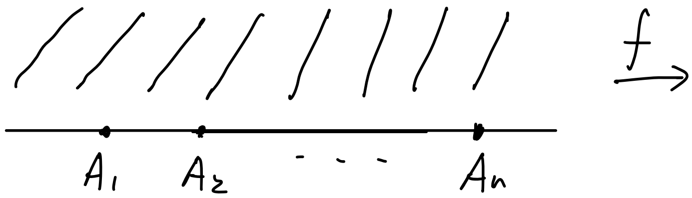
$K_1 \rightsquigarrow \{g_{n,1}(z)\}$ uniformly on K_1

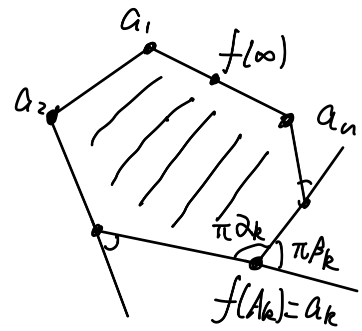
$\rightsquigarrow \{g_{n,2}(z)\}$ uniformly on K_2

$\rightsquigarrow \{g_{n,n}(z)\}$ uniformly on K_j for any $j \geq 1$.

$\blacksquare$

Conformal map to polygon.





$$\left(\begin{array}{l} \sum_{k=1}^{n} \beta_k = 2 \\ \underline{0 < \beta_k < 1} \end{array} \right)$$

$$\pi \sum_k \alpha_k = (n-2)\cdot\pi$$

$$\sum_k \underbrace{(\pi - \pi\alpha_k)}_{\pi\beta_k} = n\cdot\pi - (n-2)\pi = 2\pi$$

Schwarz–Christoffel integral

$$\boxed{S(z) = \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_n)^{\beta_n}}}$$

$$\underline{(z - A_k)^{\beta_k}} = e^{\underline{\beta_k \cdot \log(z - A_k)}} = \begin{cases} (x - A_k)^{\beta_k} & z = x > A_k \\ e^{i\pi\beta_k} \cdot (A_k - x)^{\beta_k} & z = x < A_k \end{cases}$$