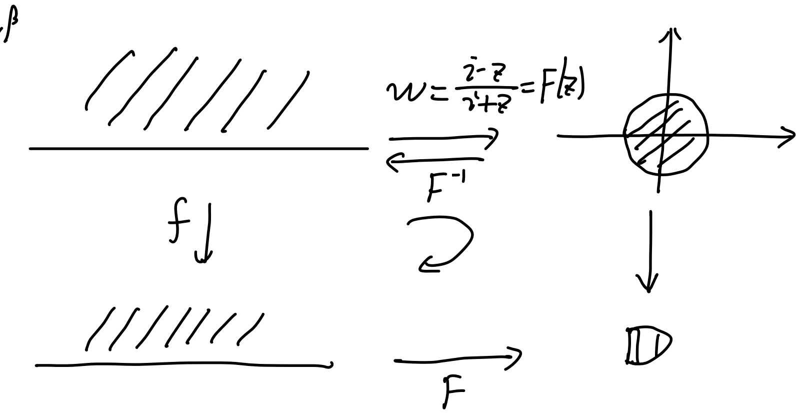
$$\mathrm{Aut}(\mathbb{D}) = \{f: \mathbb{D} \to \mathbb{D} \text{ holomorphic, bijective}\}$$

$$= \Big\{ \underbrace{e^{i\beta} \frac{a-z}{1-\bar{a}z}}_{= \varphi_{a,\beta}}, \quad a \in \mathbb{D},\ 0 \le \beta < 2\pi \Big\}.$$

$$\mathrm{Aut}(\mathbb{H}) = \{ F^{-1} \circ \varphi_{a,\beta} \circ F \}$$



$$= \boxed{\Big\{ \frac{az+b}{cz+d} : ad-bc > 0 \Big\}} \quad a,b,c,d \in \mathbb{R}$$

$$\mathrm{Aut}(\mathbb{D}) \ni F \circ f \circ F^{-1} = \varphi_{a,\beta} \Leftrightarrow f = F^{-1} \circ \varphi_{a,\beta} \circ F$$

$$= \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc = 1 \Big\} \Big/ \{\pm Id\} = PSL(2,\mathbb{R}) \quad a,b,c,d \in \mathbb{R}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

Step 1: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) \quad ad-bc \neq 0.$ $f_M(z) = \frac{az+b}{cz+d} = \frac{(ta)z+tb}{(tc)z+td}.$

$$= f_{tM}(z)$$

$$\underline{\mathrm{Im}(f_M(z))} = \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)$$

$$= \frac{1}{2i} \cdot \frac{(ad-bc)\cdot(z-\bar{z})}{|cz+d|^2} = \frac{(ad-bc)(\mathrm{Im}\, z)}{|cz+d|^2} \xrightarrow[\mathrm{Im}\, z > 0]{ad-bc>0} \underline{\mathrm{Im}(f_M(z)) > 0}$$

$$\Rightarrow \quad ad-bc>0 \iff f_M: \mathbb{H} \to \mathbb{H}$$

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}.$$

Step 2: $\underline{f_{M_1} \circ f_{M_2} = f_{M_1 M_2}}$ $\quad GL(2,\mathbb{R}) \to \mathrm{Aut}(\mathbb{H})$. is homo. group.

$$M \mapsto f_M$$

$$\underline{f_M \circ f_{M^{-1}} = f_{I_2} = Id.}$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \frac{1\cdot z+0}{0\cdot z+1} = z = Id(z)$$

step 3: $\underbrace{\{f_M : M \in SL(2,\mathbb{R})\}}_{\{M \in GL(2,\mathbb{R}) \,:\, \det M = 1\}}$ is a group $\quad f_M = f_{tM}$

$\det(tM) = t^2 \det(M) = 1$

$\Downarrow$

$t = \pm \frac{1}{\sqrt{\det M}}$

act transitively on $\mathbb{H}$.

$\mathbb{H} \ni \underbrace{z = a+bi}_{\beta} \quad b>0 \longrightarrow b\cdot i \xrightarrow{z \mapsto b^{-1} z} i = f_{M_2}(z)$

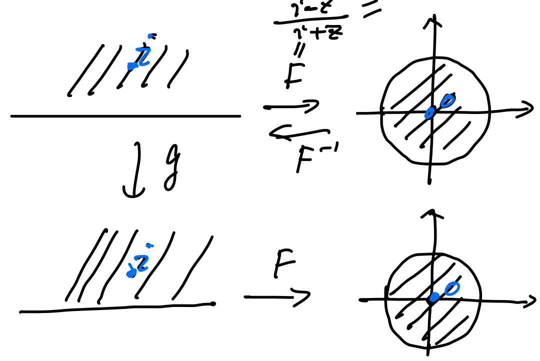
$z-a = \frac{z-a}{0\cdot z+1} \leftrightsquigarrow \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$

$M_2 = \begin{pmatrix} b^{-\frac12} & 0 \\ 0 & b^{\frac12} \end{pmatrix} \in SL(2,\mathbb{R})$

$f: \mathbb{H} \to \mathbb{H}, \quad f(\beta) = i. \quad \exists\, f_N(i) = \beta \quad g = \underbrace{f\circ f_N}_{f_M}(i) = i$

$\underbrace{F\circ g \circ F^{-1}}_{h \,\in\, \mathrm{Aut}(\mathbb{D})}(0) = 0.$

$F = \frac{i-z}{i+z} = f_{\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}} = M$



Schwarz lemma: $h: \mathbb{D} \to \mathbb{D} \quad h(0) = 0$; $\quad h^{-1}: \mathbb{D} \to \mathbb{D}$

$\Rightarrow \left.\begin{matrix} |h(z)| \le |z| \\ |h^{-1}(z)| \le |z| \end{matrix}\right\} \Rightarrow |h(z)| = |z| \Downarrow h(z) = e^{i\theta}\cdot z$

$\Rightarrow F\circ g\circ \underbrace{F^{-1}(w)}_{z}, \; w \in \mathbb{D}, \quad = e^{i\theta}\cdot \underbrace{w}_{F(z)} \Rightarrow g(z) = F^{-1}(e^{i\theta}\cdot F(z))$

$F\circ g(z) = e^{i\theta}\cdot F(z)$

$f_{M^{-1}} \circ f_{\begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}} \circ f_M = f_{M^{-1}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} M}$

$-i\left(e^{\frac{i\theta}{2}} e^{-\frac{i\theta}{2}}\right) = -i e^{\frac{i\theta}{2}} + i e^{-\frac{i\theta}{2}}$

$M = \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} \quad M^{-1} = \frac{1}{-2i}\begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix}.$

$\frac{i}{2}\begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix}\begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -i & -i \end{pmatrix}\cdot\begin{pmatrix} -e^{\frac{i\theta}{2}} & i e^{\frac{i\theta}{2}} \\ e^{-\frac{i\theta}{2}} & i e^{-\frac{i\theta}{2}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}} & \\ i e^{-\frac{i\theta}{2}} - i e^{\frac{i\theta}{2}} & \end{pmatrix}$

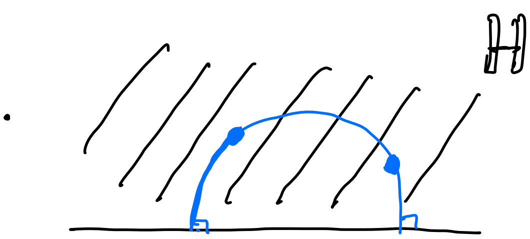
$e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}$

$SL(2,\mathbb{R}) \ni \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = M_{\frac{\theta}{2}}$

$$g(z) = f_{M_{\frac{\theta}{2}}}(z) = f \circ f_{\underset{\substack{\cap \\ SL(2,\mathbb{R})}}{N}} \Rightarrow f = f_{M_{\frac{\theta}{2}}} \circ f_{N^{-1}} = f_{\underset{\substack{\cap \\ SL(2,\mathbb{R})}}{M_{\frac{\theta}{2}} \cdot N^{-1}}}$$

∎

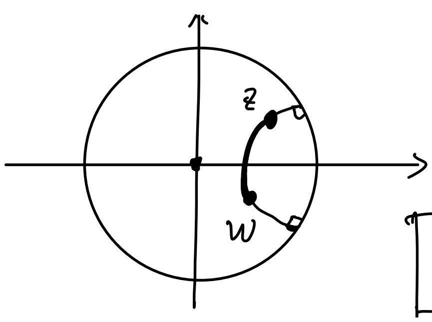
$$\mathrm{Aut}(\mathbb{H}) \ni f = f_M \quad M \in SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc=1 \right\}.$$



$$\underset{\substack{\| \\ \gamma(t) = (x(t),\, y(t))}}{L(\gamma)} = \int_\gamma \frac{ds}{y} = \int \frac{\sqrt{x'^2+y'^2}\,dt}{y(t)}.$$

$$\frac{ds^2}{y^2} = \frac{|dz|^2}{(\mathrm{Im}\, z)^2}.$$

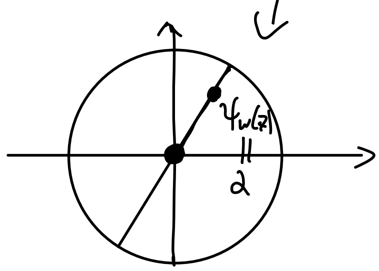
$\updownarrow$



$$\underline{\frac{ds^2}{(1-|z|^2)^2}} \qquad \int \frac{ds}{1-|z|^2}$$

$$\boxed{\mathrm{Aut}(\mathbb{D}) = \{\varphi_{\alpha,\beta}\} = \mathrm{Isom}(\mathbb{D}, g_p)}$$

$$\varphi_w(w) = 0 \qquad \varphi_w(z) = \frac{w-z}{1-\bar{w}z}$$



$$\int_0^{|a|} \frac{dt}{1-t^2} = \int_0^{|a|} \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt = \frac{1}{2} \ln \frac{1+t}{1-t}$$

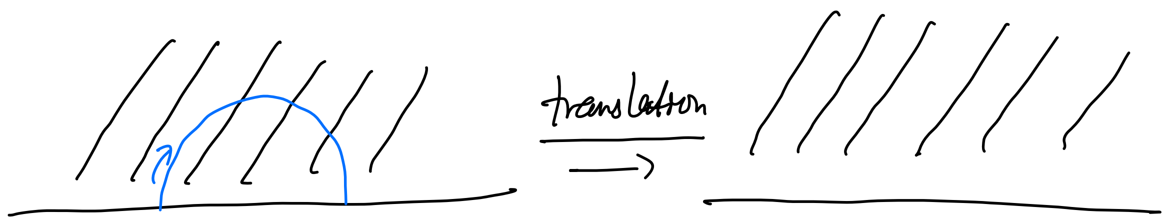
$$= \frac{1}{2} \log \frac{1+|a|}{1-|a|}$$

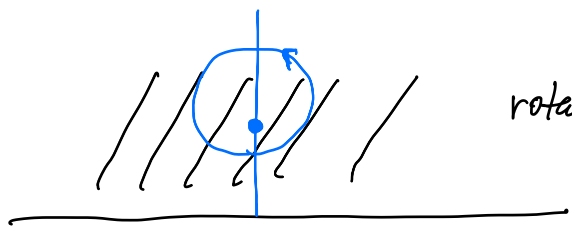
$f: \mathbb{D} \to \mathbb{D}$. $\qquad d_p(z,w) \geq d_p(f(z), f(w))$

$$\left|\frac{w-z}{1-\bar{w}z}\right| \geq \left|\frac{f(w)-f(z)}{1-\overline{f(w)}f(z)}\right|$$

"=" holds for all w and z $\iff$ f is automorphism.
$$f = \psi_{\alpha,\beta}.$$

- $\mathrm{Aut}(\mathbb{H}) = \left\{ \dfrac{az+b}{cz+d} : \begin{array}{c} ad-bc>0 \\ a,b,c,d \in \mathbb{R} \end{array} \right\}$.





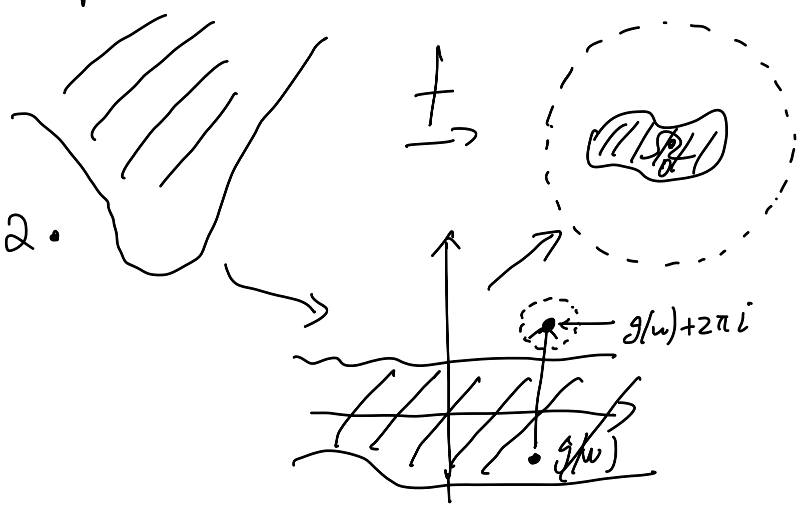
rotation.

Thm (Riemann Mapping Thm) If $U \subsetneq \mathbb{C}$ is simply connected

Then U is conformally equivalent to $\mathbb{D}$.

$\exists f: U \to \mathbb{D}$, bijective holomorphic.

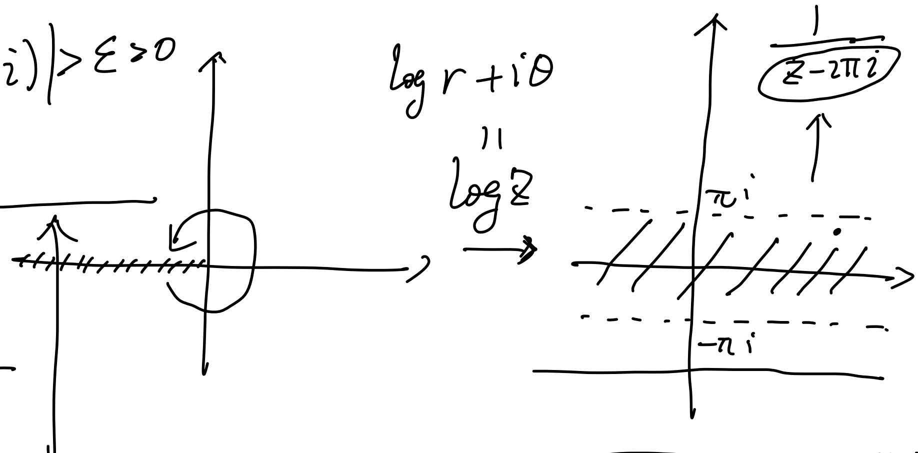
Idea of proof: step 1: $\exists f: U \to f(U) \subset \mathbb{D}$ conformal



$\underline{\log(z-a) = g}$ $\quad e^{g(z)} = z-a \quad (g(z_1) = g(z_2) \Rightarrow z_1 = z_2)$

$\exists \varepsilon > 0$, $|g(z) - (g(w) + 2\pi i)| > \varepsilon > 0$

$\forall z \in U$.



$$\left| \frac{1}{g(z) - (g(w) + 2\pi i)} \right| \leq \frac{1}{\varepsilon}$$

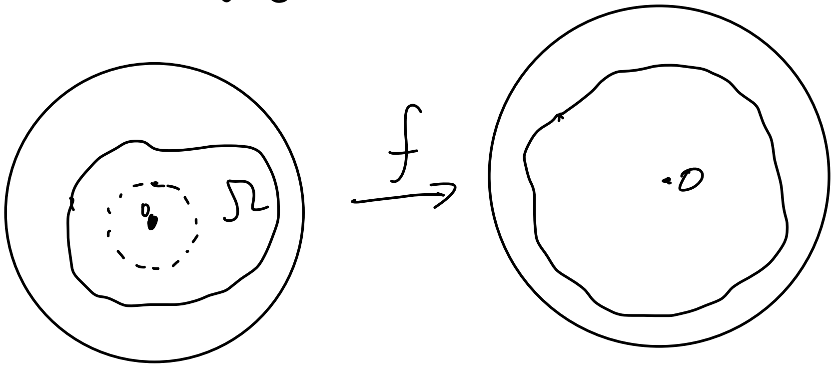
If not true, then $\exists z_n$ s.t. $e^{g(z_n)} \to e^{g(w) + 2\pi i}$

$z_n - a \to w - a \Rightarrow z_n \to w$

$g(z_n) \to g(w)$ $\Uparrow$

Step 2: $\mathcal{F} = \left\{ f : \overset{0}{\underset{\cap}{}}\Omega \to \mathbb{D} \ \ \text{injective holomorphic},\ f(0)=0 \right\}$

$$s = \sup_{f\in\mathcal{F}} |f'(0)| \le M < +\infty .$$



$$|f'(0)| \le \frac{1}{2\pi}\int_{C_\varepsilon} \frac{|f(\zeta)|}{|(\zeta-0)^2|}\,|d\zeta| = \frac{1}{2\pi\varepsilon^2}\cdot 1\cdot 2\varepsilon = \frac{1}{2\pi\varepsilon} < +\infty$$

$\Rightarrow \exists \{f_n\} \subset \mathcal{F}$ s.t. $|f_n'(0)| \to s$.

Want take limit: $\exists$ subsequence $\{f_{n_k}\}$ s.t.

$\boxed{\lim_{k\to+\infty} f_{n_k}(z) = f(z) \text{ is holomorphic}}$ ← Montel's Thm.

and $\boxed{f \text{ is conformal equivalence from } \Omega \text{ to } \mathbb{D}.}$

1. f is injective $\qquad f'(0) = s$

2. f is surjective.

Proof of surjectivity: Proof by contradiction.

suppose $f(z) \neq 2 \quad \forall z \in \Omega$.