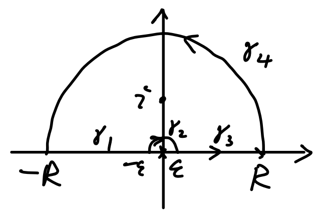
$$\int_{-\infty}^{+\infty} \frac{\sin(x)}{x\cdot(1+x^2)^2}\,dx$$



$$f(z)=\frac{e^{iz}}{z(1+z^2)^2}=\frac{e^{iz}}{z(z+i)^2(z-i)^2}$$

$$\int_\gamma f(z)dz = 2\pi i\cdot \mathrm{res}_{z=i} f(z) = 2\pi i\cdot\left(-\frac{3}{4}e^{-1}\right).$$

$$\mathrm{res}_{z=i} f(z) = \frac{d}{dz}\left((z-i)^2 f(z)\right)\Big|_{z=i} = \frac{d}{dz}\left(\frac{e^{iz}}{z(z+i)^2}\right)\Big|_{z=i}$$

$z(z+i)^2$
$\shortparallel$
$z(z^2+2iz-1)$
$\shortparallel$
$z^3+2iz^2-z.$

$$=\frac{e^{iz}\cdot i}{z(z+i)^2} - \frac{e^{iz}}{z^2(z+i)^4}\cdot(3z^2+4iz-1)\Big|_{z=i}$$

$$=\frac{e^{-1}\cdot \cancel{i}}{\cancel{i}\cdot(2i)^2} - \frac{e^{-1}}{i^2(2i)^4}(3\cdot i^2+4i\cdot i-1)$$

$$=-\frac{e^{-1}}{4} - \frac{e^{-1}}{\cancel{-16}}\cdot(-3-4-1)^{-8} = -\left(\frac{1}{4}+\frac{1}{2}\right)e^{-1} = -\frac{3}{4}e^{-1}.$$

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{-\varepsilon}\frac{e^{ix}}{x(x^2+1)^2}dx = \int_{-R}^{-\varepsilon}\frac{\cos x + i\sin x}{x(x^2+1)^2}dx = -\int_\varepsilon^R \frac{\cos x}{x(x^2+1)^2}dx + i\int_{-R}^{-\varepsilon}\frac{\sin x}{x(x^2+1)^2}dx$$

$$\int_{\gamma_2} f(z)dz = \int_\pi^0 \frac{e^{i\cdot\varepsilon e^{i\theta}}}{\cancel{\varepsilon e^{i\theta}}\cdot(1+\varepsilon^2 e^{2i\theta})^2}\,\varepsilon \cancel{e^{i\theta}}\, i\,d\theta = -i\int_0^\pi \frac{e^{i\cdot\varepsilon e^{i\theta}}}{(1+\varepsilon^2e^{2i\theta})^2}\,d\theta \to -i\int_0^\pi 1\,d\theta$$

$\shortparallel$
$-i\pi.$

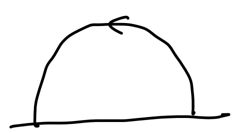
$\downarrow \varepsilon\to 0$

$$\frac{e^0}{(1+0^2)^2}=1$$

$z=\varepsilon\cdot e^{i\theta}$, $dz=\varepsilon e^{i\theta}\cdot i\,d\theta$

$\theta:\pi\to 0$

$$\int_{\gamma_3} f(z)dz = \int_\varepsilon^R \frac{\cos x}{x(1+x^2)^2}dx + i\cdot\int_\varepsilon^R\frac{\sin x}{x(x^2+1)^2}dx$$



$$\int_{\gamma_4} f(z)dz = \int_0^\pi \frac{e^{i\cdot Re^{i\theta}}}{Re^{i\theta}(1+R^2e^{2i\theta})^2}\,Re^{i\theta} i\,d\theta \xrightarrow{R\to+\infty} 0$$

$\le \frac{1}{(R^2-1)^2}$ length $\sim \pi R$

$z(\theta)=R\cdot e^{i\theta}$

$e^{iR(\cos\theta+i\sin\theta)}$
$\shortparallel$
$e^{-R\sin\theta}\, e^{iR\cos\theta}$

$$i\left(\int_{-R}^{-\varepsilon}+\int_\varepsilon^R\right)\frac{\sin x}{x(1+x^2)^2}dx \;\; - i\pi = 2\pi i\cdot\left(-\frac{3}{4}e^{-1}\right)$$

$+$
$\int_{\gamma_4}$

$\shortparallel$
$\int_{\gamma_2}$

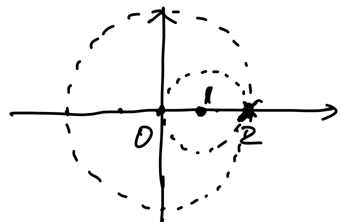
$\varepsilon \to 0$
$R \to +\infty$

$$\cancel{x}\int_{-\infty}^{+\infty} \frac{\sin x}{x(1+x^2)^2}\,dx = \cancel{x}\pi - \pi\cancel{x}\cdot\frac{3}{2}e^{-1}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\sin x}{x(1+x^2)^2}\,dx = \pi - \frac{3}{2}\pi e^{-1}.$$

$$\frac{e^{iz}}{z\cdot(1+z^2)^2} = \frac{\overbrace{1+iz+\frac{(iz)^2}{2}+\cdots}^{\to 1}}{\underbrace{z(1+z^2)^2}_{\to 0}} \qquad \operatorname{res}_{z=0} f(z) = 1.$$

$$\frac{z}{(z-2)^2} = z\cdot\frac{1}{(z-2)^2}$$



$$\frac{1}{z-2} = -\frac{1}{2(1-\frac{z}{2})} = -\frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{+\infty}\frac{z^n}{2^{n+1}}.$$

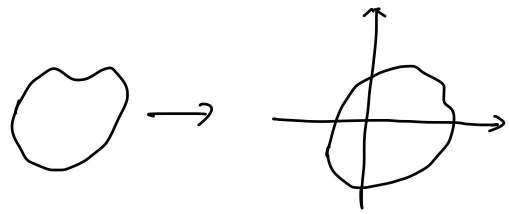
$$\frac{d}{dz}\frac{1}{z-2} = -\sum_{n=0}^{+\infty}\frac{n\cdot z^{n-1}}{2^{n+1}} \Rightarrow \frac{1}{(z-2)^2} = \sum_{n=1}^{+\infty}\frac{n\cdot z^{n-1}}{2^{n+1}}$$

$$\frac{d}{dz}\frac{1}{z-2} = -\frac{1}{(z-2)^2}$$

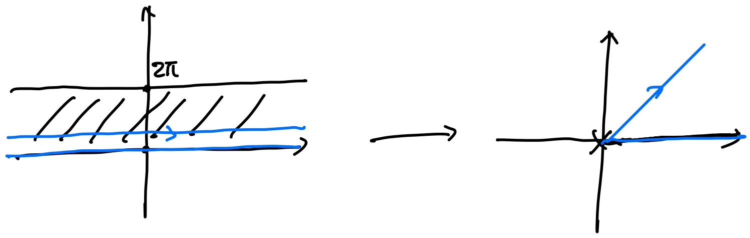
$$\frac{z}{(z-2)^2} = \sum_{n=1}^{+\infty}\frac{n\cdot z^n}{2^{n+1}}$$

$$\left(\frac{z}{(z-2)^2} = \frac{z-1+1}{((z-1)-1)^2} \overset{w=z-1}{=\!=} \frac{1+w}{(w-1)^2} = \frac{1}{(1-w)^2} + \frac{w}{(1-w)^2} \qquad |w|=|z-1|<1\right)$$

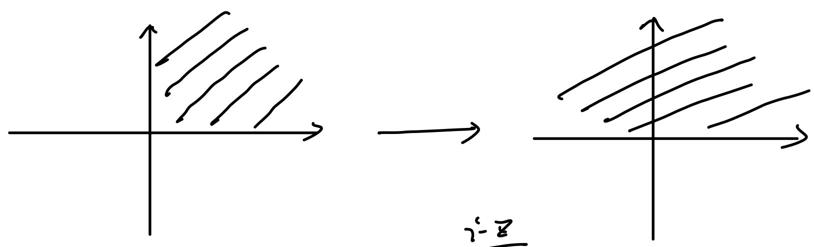
Conformal mappings: $z \mapsto f(z)$



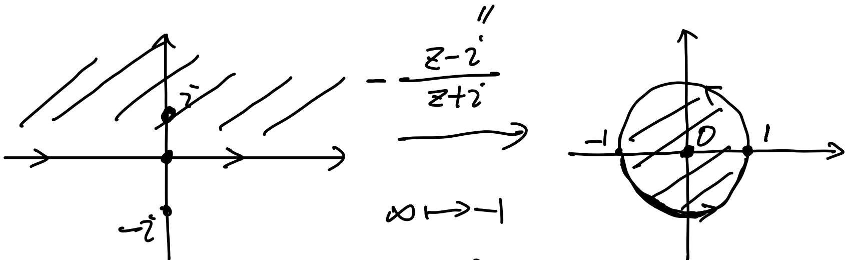
e^z



z^2
$=$
$r^2 e^{2i\theta}$



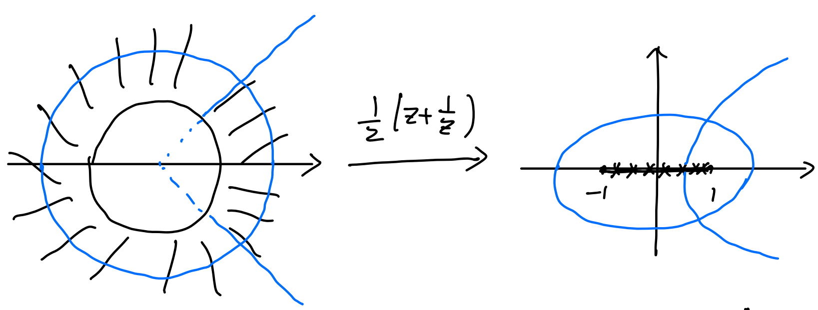
$$\frac{i-z}{i+z} = -\frac{z-i}{z+i}$$



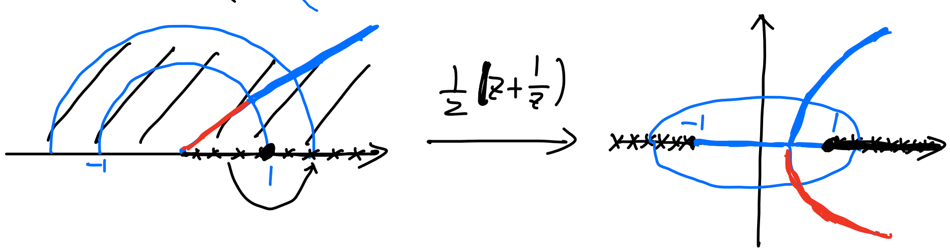
$\infty \mapsto -1$

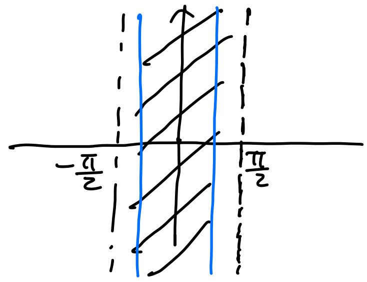
$$\frac{i-x}{i+x} = -\frac{(x-i)(x-i)}{(x+i)(x-i)} = \frac{-(x^2-1-2ix)}{x^2+1} = \frac{1-x^2+i\,2x}{x^2+1}$$

$\frac{1}{2}\left(z+\frac{1}{z}\right)$

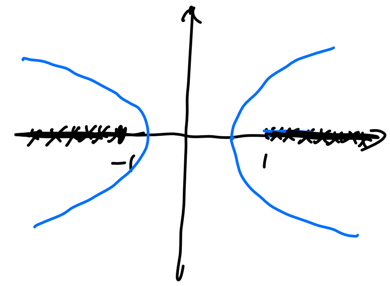


$\frac{1}{2}\left(z+\frac{1}{z}\right)$

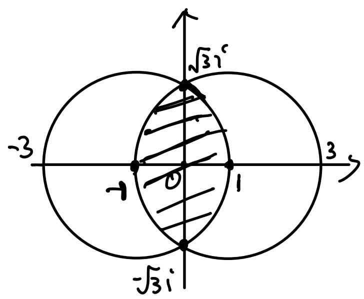




$\xrightarrow{\sin(z)}$



Ex:



$$z_1 = \frac{z-\sqrt{3}i}{z+\sqrt{3}i}$$

$\sqrt{3}i \mapsto 0$

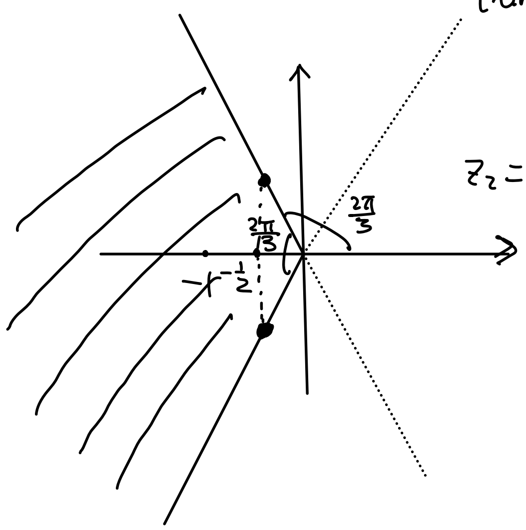
$-\sqrt{3}i \mapsto \infty \in \mathbb{C}\cup\{\infty\}.$

$-1 \mapsto \frac{-1-\sqrt{3}i}{-1+\sqrt{3}i} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i.$

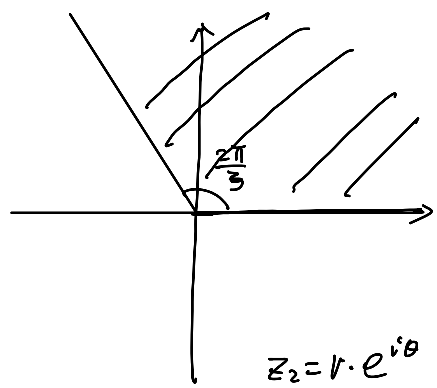
$1 \mapsto \frac{1-\sqrt{3}i}{1+\sqrt{3}i} = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$0 \mapsto -1.$

Fact: $\frac{az+b}{cz+d}$ maps circles (lines) to circles (lines)



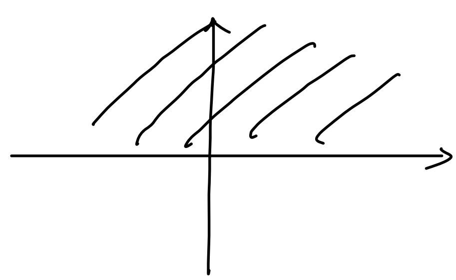
$z_2 = e^{-\frac{2\pi}{3}i}\cdot z_1$ $\longrightarrow$



$z_2 = r\cdot e^{i\theta}$

$z_3 = r^{\frac{3}{2}} e^{i\theta\cdot\frac{3}{2}} = z_2^{\frac{3}{2}}$

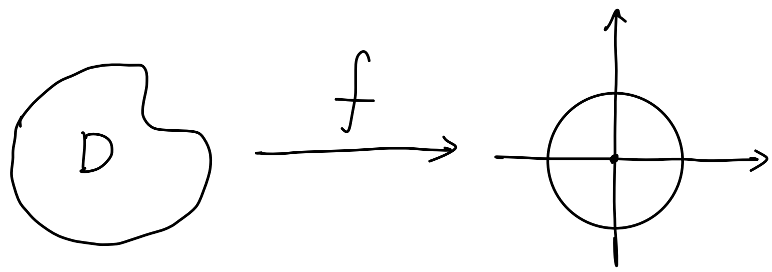
$\downarrow$



$$z \mapsto z_1 \mapsto z_2 \mapsto z_3 = z_2^{\frac{3}{2}} = \left(e^{-\frac{2\pi}{3}i}\cdot z_1\right)^{\frac{3}{2}}$$

$$= \underbrace{e^{-\pi i}}_{=-1}\left(\frac{z-\sqrt{3}i}{z+\sqrt{3}i}\right)^{\frac{3}{2}}$$

Thm (Riemann Mapping Thm) Any simply connected domain U in $\mathbb{C}$ that is not $\mathbb{C}$ is conformally equivalent to the unit disk.



Schwarz Lemma: $f: \mathbb{D}=\{|z|<1\} \longrightarrow \mathbb{D}$, $f(0)=0$

Then:
- $|f(z)| \le |z|$. for any $z \in \mathbb{D}$
- If for some $z_0 \ne 0$, $|f(z_0)| = |z_0|$, then $f(z)=e^{i\beta}\cdot z$ (rotation)
- $|f'(0)| \le 1$. if equality holds. then $f(z)=e^{i\beta}\cdot z$ (a rotation).

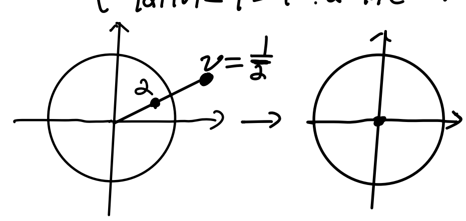
Pf: $g(z)=\frac{f(z)}{z}$ is holomorphic in $\mathbb{D}$. Apply maximum principle. ∎

- Automorphism group of $\mathbb{D}$: $\mathrm{Aut}(\mathbb{D}) = \left\{ f: \mathbb{D}\to\mathbb{D}.\ \text{holomorphic bijective} \right\}$.

$$\begin{cases} \cdot\ f(z)=z \text{ is in } \mathrm{Aut}(\mathbb{D}) \\ \cdot\ f, g \in \mathrm{Aut}(\mathbb{D}),\ g\circ f \in \mathrm{Aut}(\mathbb{D}) \\ \cdot\ f \in \mathrm{Aut}(\mathbb{D}),\ f^{-1} \in \mathrm{Aut}(\mathbb{D}). \end{cases}$$

$f^{-1} \ne \frac{1}{f}$

$\left(|v| = \frac{1}{|a|},\ v = \frac{1}{|a|}\cdot e^{i\theta} = \frac{1}{\bar a} \quad |a|\cdot|v| = 1^2 = 1.\ \bar a = |a| e^{-i\theta} \right)$



$\mathrm{Aut}(\mathbb{D}) \ni f: \mathbb{D} \longrightarrow \mathbb{D}$

$a \longmapsto \underline{0}$

$\underset{\mathrm{Aut}(\mathbb{D})\ \ni}{\psi_a}(z) = \frac{z-a}{z-\frac{1}{\bar a}}\left(\frac{+1}{\bar a}\right) = \frac{a-z}{1-\bar a\cdot z} : \mathbb{D}\to\mathbb{D}$

$g = f\circ \varphi_a^{-1} : \mathbb{D}\to\mathbb{D}$ $\qquad$ $\underline{g(0)=0}$.

$0\mapsto a\mapsto 0$

Schwarz Lemma $\Rightarrow$ $\boxed{|g(z)|\le|z|.\ \text{for all } z\in\mathbb{D}. \qquad |g(z)|\ge|z|,\ \forall z.}$

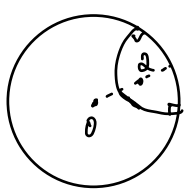
$\Updownarrow$

$g^{-1}:\mathbb{D}\to\mathbb{D},\quad g^{-1}(0)=0.$ $\xRightarrow{\text{Schwarz Lemma}}$ $|\underbrace{g^{-1}(z)}_{z}|\le|\underbrace{z}_{g(z)}|\quad \forall z$

$\Rightarrow |g(z)|=|z|$, for all $z\in\mathbb{D}$ $\xRightarrow{\text{Schwarz}}$ $g(z)=e^{i\beta}\cdot z$.

$f\circ \underbrace{\varphi_a^{-1}(z)}_{z} = e^{i\beta}\cdot \underbrace{z}_{\varphi_a(z)} \Rightarrow f(z)=e^{i\beta}\cdot\varphi_a(z).$

$\varphi_a = \dfrac{a-z}{1-\bar{a}\cdot z}$ $\quad a\mapsto 0,\ 0\mapsto a$, $\quad \varphi_a\circ\varphi_a(z)= z.\quad \varphi_a=\varphi_a^{-1}$



$$\boxed{\mathrm{Aut}(\mathbb{D}) = \left\{ e^{i\beta}\cdot\varphi_a(z),\ a\in\mathbb{D}\right\} \quad \varphi_a(z) = e^{i\beta}\cdot\frac{a-z}{1-\bar{a}z}}$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{D} \arrow[r, shift left, "F^{-1}(w) = i\frac{1-w}{1+w}"] \arrow[d, "f"'] & \mathbb{H} \arrow[l, shift left, "F"] \arrow[d, "\varphi"] \\
\mathbb{D} \arrow[r, shift right, "F"'] & \mathbb{H} \arrow[l, shift right, "F^{-1}"']
\end{tikzcd}
$$

$$w = \frac{i-z}{i+z}$$

$$wi + w\cdot z = i - z$$

$$(w+1)z = i - wi$$

$$f = F \circ \varphi \circ F^{-1} \in Aut(\mathbb{D}). \Rightarrow \varphi = F^{-1} \circ f \circ F$$

$$Aut(\mathbb{H}) = \{ F^{-1} \circ f \circ F : f \in Aut(\mathbb{D}) \}.$$

$$= \left\{ \frac{az+b}{cz+d} : \begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc > 0 \end{array} \right\}$$

$$\frac{(-a)\cdot z + (-b)}{(-c)\cdot z + (-d)} \quad \frac{(t\cdot a)z + t\cdot b}{(t\cdot c)z + t\cdot d} \quad t \in \mathbb{R},\ t \neq 0$$

$$Aut(\mathbb{H}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim t\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \ t \neq 0 \right\} = PSL(2, \mathbb{R})$$