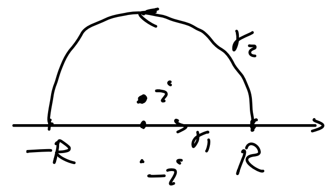
$\xi\in\mathbb{R}_{>0}$

$$\int_{-\infty}^{+\infty}\frac{e^{-2\pi i x\cdot\xi}}{(1+x^2)^2}dx \qquad f(z)=\frac{e^{2\pi i\xi\cdot z}}{(1+z^2)^2}=\frac{e^{2\pi i\xi\cdot z}}{(z+i)^2(z-i)^2}=\frac{g(z)}{(z-i)^2}$$

$$\parallel$$

$$\frac{\cos(2\pi x\xi)}{(1+x^2)^2}+\frac{i\cdot\sin(2\pi x\xi)}{(1+x^2)^2}$$



$\int_\gamma f(z)dz = 2\pi i\cdot res_{z=i} f$ $\qquad$ i is a pole of order 2

$$\Rightarrow res_{z=i} f=\frac{d}{dz}\Big|_{z=i}\left((z-i)^2 f\right)=\frac{d}{dz}\frac{e^{2\pi i\xi\cdot z}}{(z+i)^2}\Big|_{z=i}=\frac{e^{2\pi i\xi\cdot z}\cdot(2\pi i\xi)}{(z+i)^2}-\frac{2\cdot e^{2\pi i\xi\cdot z}}{(z+i)^3}\Big|_{z=i}$$

$$=\frac{e^{-2\pi\xi}2\pi i\xi}{-4}+\frac{8\cdot e^{-2\pi\xi}}{4+8i}=-\frac{1}{4}\cdot e^{-2\pi\xi}\cdot(2\pi\xi+1)\cdot i$$

$$res_{z=z_0} f=\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n\cdot f(z)\Big|_{z=z_0}$$

$$\int_{\gamma_1} f(z)dz=\int_{-R}^{R}\frac{e^{-2\pi i x\xi}}{(1+x^2)^2}dx\longrightarrow I. \qquad f(z)=\frac{e^{2\pi i\xi\cdot z}}{(1+z^2)^2}$$

$$\left|\int_{\gamma_2} f(z)dz\right|\le\int_{\gamma_2}|f(z)|\,|dz|\le \pi\cdot R\ \max_{\gamma_2}|f(z)|\le\frac{C}{R}\xrightarrow{R\to+\infty}0$$

$$\parallel$$

$R\cdot e^{i\theta}$, $0\le\theta\le\pi$

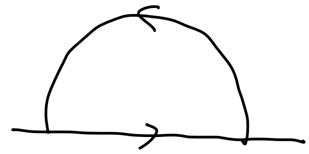
$$\left|\frac{e^{2\pi i\cdot\xi\cdot R(\cos\theta+i\cdot\sin\theta)}}{(1+R^2e^{2i\theta})^2}\right|=\frac{|e^{2\pi\xi R\cdot(-\sin\theta+i\cos\theta)}|}{|(1+R^2e^{2i\theta})^2|}\le\frac{e^{-2\pi\xi R\cdot\sin\theta}}{(R^2-1)^2}$$

$$\ge$$

$$R^2-1$$

$$I=2\pi i\cdot res_{z=i} f=2\pi i\left(-\frac{1}{4}e^{-2\pi\xi}(2\pi\xi+1)i\right)=\frac{\pi}{2}e^{-2\pi\xi}(1+2\pi\xi)$$

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^{n+1}} = \lim_{R\to\infty} \int_{\gamma_R} \frac{dz}{(1+z^2)^{n+1}}$$



$$f(z) = \frac{1}{(1+z^2)^{n+1}} = \frac{1}{(z+i)^{n+1}\cdot(z-i)^{n+1}}$$

$$\underline{\operatorname{res}_{z=i} f(z) = \frac{1}{n!}\frac{d^n}{dz^n}\left((z-i)^{n+1}\cdot f(z)\right)\Big|_{z=i} = \frac{1}{n!}\frac{d^n}{dz^n}\frac{1}{(z+i)^{n+1}}}$$

$$= \frac{1}{n!}\cdot\frac{(-1)^n\cdot(n+1)\cdot(n+2)\cdots(2n)}{\underbrace{(z+i)^{2n+1}}_{(2i)^{2n+1}}}\Bigg|_{z=i} = \frac{1}{n!}\cdot\frac{\cancel{(-1)^n}\cdot(n+1)(n+2)\cdots(2n)}{\cancel{(-1)^n}\cdot i\cdot 2^{2n+1}}$$

$$= \frac{1}{i}\cdot\frac{1}{2^{2n+1}}\cdot\frac{(2n)!}{(n!)^2} = \frac{1}{2i}\cdot\frac{(2n)\cdot(2n-1)\cdots 2\cdot 1}{(2n)^2\cdot(2n-2)^2\cdots 2^2}$$

$$= \frac{1}{2i}\cdot\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{(2n)\cdot(2n-2)\cdots 2\cdot 1} = \frac{1}{2i}\cdot\frac{(2n-1)!!}{(2n)!!}$$

$$\Rightarrow I = \pi\cdot\frac{(2n-1)!!}{(2n)!!}.$$

$$\left(\int_0^{\pi} \log(\sin x)\cdot dz\right)\cdot\frac{1}{\pi}$$

$$\shortparallel$$

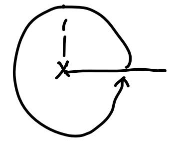
9. $\displaystyle\int_0^1 \log(\sin \pi x)\,dx$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

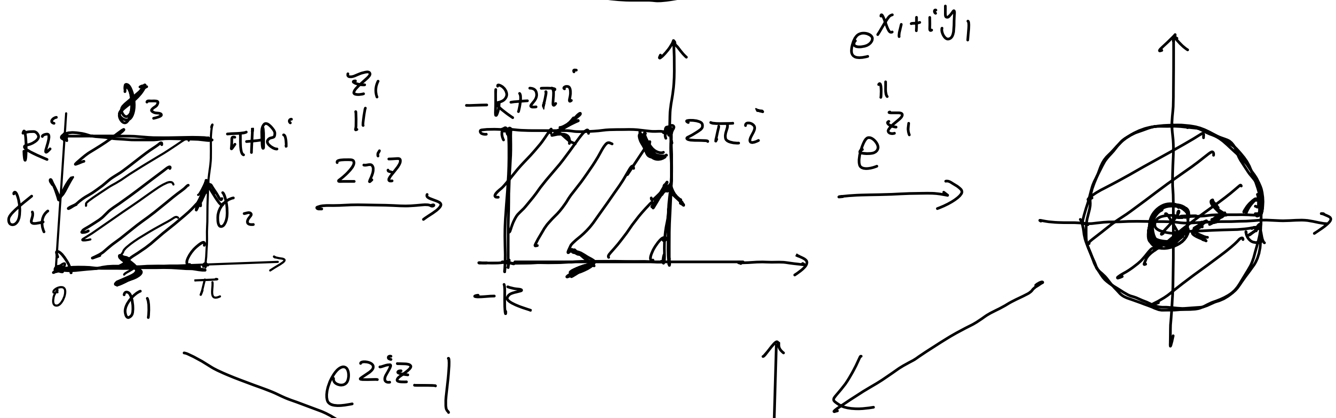
$$= \frac{e^{-iz}\cdot(e^{2iz}-1)}{2i}$$

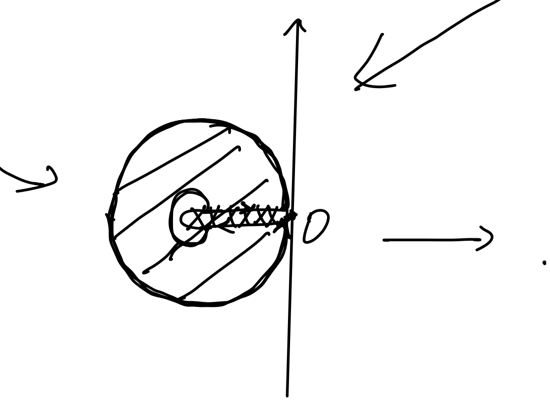
$$\log(\sin z) = \log \frac{e^{-iz}\cdot(e^{2iz}-1)}{2i}$$

$$= \log e^{-iz} + \log(e^{2iz}-1) - \log 2 - \log i$$



$$f(z) = \log(e^{2iz}-1)$$





$$0 = \int_{\gamma_R} f(z)\,dz$$

$$\int_{\gamma_1} f(z)\,dz = \int_0^{\pi} \log(e^{2ix}-1)\,dx$$

$$e^{2ix}-1 = (\cos(2x) + i\sin(2x) - 1)$$
$$= -2\cdot\sin^2 x + 2i\cdot\sin x\cdot\cos x$$
$$= 2\sin x\cdot(-\sin x + i\cdot\cos x)$$

$$\cos(2x) = 1 - 2\cdot\sin^2 x$$

$$\log(e^{2ix}-1) = \log 2 + \log\sin x + i\cdot\theta(x)$$

$$\int_0^\pi \log\sin x\,dx + (\log 2)\cdot\pi = 0$$
$$\Rightarrow \int_0^\pi \log\sin x\,dx = -\log 2\cdot\pi.$$

\# of zeros.

Ex: $z^5 + 13z^2 + 15 = 0$ in annulus $1 < |z| < 2$.

$|f| > |g| \Rightarrow f$ and $f \pm g$ have same # of zeros.

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$|z|=1$. $|z^5 + 13z^2| < |z|^5 + 13\cdot|z|^2 = 1 + 13 = 14 < 15$

$\Rightarrow z^5 + 13z^2 + 15$ and 15 have same number of zeros in $|z|<1$.

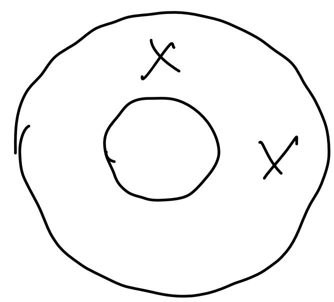
$$\underbrace{\text{same number}}_{= 0}$$

$|z|=2$, $|13\cdot z^2| = 13\cdot 2^2 = 13\cdot 4 = 52$.

$$|z^5| = 2^5 = 32$$

$$|z^5+15| \leq 32+15 = 47 < |13\cdot z^2| = 52$$

$\Rightarrow$ z^5+13z^2+15 and $13\cdot z^2$ have the same number of zeros in $|z|<2$



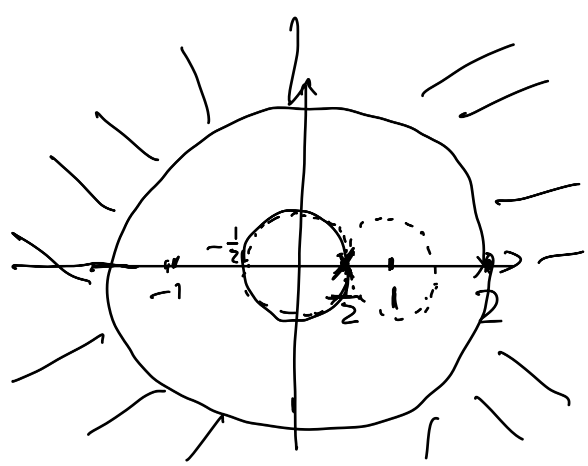
<u>Ex:</u> $\dfrac{3z}{(2-z)(2z-1)} = f(z)$. Taylor expansion near 0

$$= \frac{-3z}{(2-z)(1-2z)}$$

$$\frac{A}{2-z} + \frac{B}{1-2z} = \frac{A-2Az+2B-Bz}{(2-z)(1-2z)} = \frac{(A+2B)-(2A+B)z}{(2-z)(1-2z)}$$

$$\begin{cases} A+2B=0 \\ 2A+B=3 \end{cases} \Rightarrow A=2,\ B=-1$$

$$f(z) = \frac{2}{2-z} - \frac{1}{1-2z}$$



$$\frac{2}{2-z} = \frac{1}{1-\frac{z}{2}} = 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots = \sum_{n=0}^{\infty} 2^{-n} \cdot z^n \qquad \left|\frac{z}{2}\right| < 2$$

$$\frac{1}{1-2z} = 1 + (2z) + \cdots = \sum_{n=0}^{\infty} (2z)^n . \qquad |2z| < 1$$

$$f(z) = \sum_{n=0}^{\infty} \left(2^{-n} - 2^n\right) z^n .$$

radius of convergence: $\left|\left(2^{-n} - 2^n\right)\right|^{\frac{1}{n}} \xrightarrow{n \to +\infty} 2$

$$R = \frac{1}{2} .$$

Laurent Series: $f(z) = \dfrac{2}{2-z} - \dfrac{1}{1-2z}$. $\quad \frac{1}{2} < |z| < 2$

$|2z| > 1$

$$\frac{1}{1-2z} = -\frac{1}{2z \cdot \left(1 - \frac{1}{2z}\right)} = -\frac{1}{2z} \sum_{n=0}^{\infty} \frac{1}{(2z)^n} \qquad \frac{1}{|2z|} < 1$$

$$f(z) = \sum_{n=0}^{\infty} 2^{-n} \cdot z^n - \sum_{n=1}^{\infty} \frac{1}{2^{n+1} \cdot z^{n+1}} \quad \text{(Laurent series)}$$

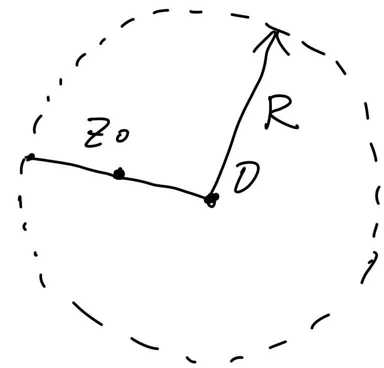
15. (a) f is an entire fct. for some k, A, B.

$$\sup_{|z|=R} |f(z)| \le A\cdot R^k + B. \text{ for all } R$$

Prove f is a polynomial.

Proof: f is a polynomial of degree $\le k \iff f^{(k+1)} \equiv 0$

$$\left|f^{(k+1)}(z_0)\right| = \left|\frac{(k+1)!}{2\pi i}\int_{|z|=R} \frac{f(\zeta)}{(\zeta - z_0)^{k+2}}\, d\zeta\right|$$



$$\le \frac{(k+1)!}{\cancel{2\pi}} \cdot \frac{A\cdot R^k + B}{(R-|z_0|)^{k+2}} \cancel{2\pi} R \xrightarrow{R\to+\infty} 0. \quad \checkmark$$

22. There are no holomorphic function f in unit disk $\mathbb{D}$ that extends continuously to $\partial\mathbb{D}$ s.t. $f(z) = \frac{1}{z}$ for $z \in \partial\mathbb{D}$

Pf: $0 = \int_{|z|=1} f(z)\,dz = \int_{|z|=1} \frac{1}{z}\,dz = 2\pi i$. $\rightarrow\leftarrow$