1.9 Cauchy-Riemann equation: $f = u + iv$ is holomorphic iff

$$0 = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\sqrt{-1}}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\Leftrightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \text{(CR eq.)}. \Leftrightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta \\ \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta \\ \frac{\partial \theta}{\partial x} = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2} = \frac{1}{r}\cdot(-\sin\theta) \\ \frac{\partial \theta}{\partial y} = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2} = \frac{1}{r}\cdot\cos\theta \end{cases} \Leftrightarrow \underbrace{\frac{\partial(r,\theta)}{\partial(x,y)}}_{\parallel\ J} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

$$J^{-1} = r \cdot \begin{pmatrix} \frac{\cos\theta}{r} & -\sin\theta \\ \frac{\sin\theta}{r} & \cos\theta \end{pmatrix}.$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = u_r \cdot \cos\theta + u_\theta \cdot \frac{1}{r}(-\sin\theta) = \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = u_r \cdot \sin\theta + u_\theta \cdot \frac{1}{r}\cdot\cos\theta = \begin{pmatrix} \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix}.$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = J^t \cdot \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} \Leftrightarrow \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \underbrace{(J^t)^{-1} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot J^t}_{\parallel} \cdot \begin{pmatrix} v_r \\ v_\theta \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot J^t \begin{pmatrix} v_r \\ v_\theta \end{pmatrix}$$

$$r\cdot\begin{pmatrix}\frac{\cos\theta}{r} & \frac{\sin\theta}{r}\\ -\sin\theta & \cos\theta\end{pmatrix}\cdot\begin{pmatrix}0&1\\-1&0\end{pmatrix}\cdot\begin{pmatrix}\cos\theta & -\frac{\sin\theta}{r}\\ \sin\theta & \frac{\cos\theta}{r}\end{pmatrix}$$

$$= \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\sin\theta & \frac{\cos\theta}{r}\\ -\cos\theta & \frac{\sin\theta}{r}\end{pmatrix} = \begin{pmatrix}0&\frac{1}{r}\\-r&0\end{pmatrix}.$$

$$\begin{pmatrix}u_r\\ u_\theta\end{pmatrix} = \begin{pmatrix}0&\frac{1}{r}\\-r&0\end{pmatrix}\begin{pmatrix}v_r\\ v_\theta\end{pmatrix} \Longleftrightarrow \begin{cases} u_r = \frac{1}{r}v_\theta \\ u_\theta = -r\cdot v_r\end{cases}$$

$$\Big\Updownarrow$$

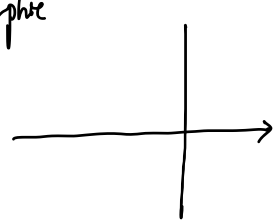
$$f \text{ is hol.} \Longleftrightarrow \frac{\partial f}{\partial \bar z} = 0 \Longleftrightarrow \bar z\,\frac{\partial f}{\partial \bar z} = 0$$

$$\log z = \log r + i\theta \qquad \bar z\,\frac{\partial f}{\partial \bar z} = \frac{\partial f}{\partial(\overline{\log z})} = \frac{1}{2}\left(\frac{\partial}{\partial \log r} + i\frac{\partial}{\partial\theta}\right)(u+iv)$$

$$= \frac{1}{2}\left(\underbrace{\frac{\partial u}{\partial\log r}}_{=\,r\frac{\partial u}{\partial r}} - \frac{\partial v}{\partial\theta}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial\theta} + \underbrace{\frac{\partial v}{\partial\log r}}_{=\,r\frac{\partial v}{\partial r}}\right)$$

1.13. $\cdot$ $\underbrace{Re(f)}_{=u} = \text{const.} \Rightarrow 0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $0 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow v$ is const.

f holomorphic $\qquad \Rightarrow f = \text{const.}$



- $\mathrm{Im}(f) = \text{const.} \Rightarrow f = \text{const.}$
- $|f|^2 = \text{const}$ $\quad \frac{\partial f}{\partial \bar z} = 0$, $\quad \frac{\partial f}{\partial z} = \overline{\frac{\partial \bar f}{\partial \bar z}} = \overline{\left(\frac{\partial (\frac{c}{f})}{\partial \bar z}\right)} = 0$

$$\underset{u^2+v^2}{\overset{\|}{}} = f \cdot \bar f \Rightarrow \bar f = \frac{c}{f} \text{ holomorphic}$$

$$\Rightarrow \frac{\partial f}{\partial \bar z} = 0 = \frac{\partial f}{\partial z} \Rightarrow \frac{\partial f}{\partial x} = 0 \ \& \ \frac{\partial f}{\partial y} = 0 \Rightarrow f \equiv \text{const.}$$

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radius of convergence: $\sum_{n=1}^{\infty} a_n \cdot z^n$

$$\overline{\lim_{n\to+\infty}} |a_n|^{\frac{1}{n}} = L \ , \quad R = \frac{1}{L}$$

1.16

- $a_n = (\log n)^2$, $\quad \log |a_n|^{\frac{1}{n}} = \frac{1}{n} \cdot \log a_n = \frac{1}{n} 2 \cdot \log\log n \xrightarrow{n\to+\infty} 0$

$$\Rightarrow |a_n|^{\frac{1}{n}} \longrightarrow 1 = e^0 \Rightarrow R = \frac{1}{1} = 1 .$$

- $a_n = n!$. $\quad n! = \underbrace{n \cdot (n-1) \cdot \ldots}_{\frac{n}{2}} \cdot 2 \cdot 1 \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}$

$c \cdot n^{n+\frac{1}{2}} e^{-n}$ Stirling

$$n!^{\frac{1}{n}} \geq \left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)^{\frac{1}{n}} = \left(\frac{n}{2}\right)^{\frac{1}{2}} \xrightarrow{n\to+\infty} +\infty = L$$

$$\Rightarrow R = \frac{1}{\infty} = 0 . \quad \sum_{n=1}^{\infty} n! \cdot z^n \text{ diverges if } z \neq 0 .$$

1.19. $\sum_{n=1}^{\infty} n\cdot z^n$ $\qquad R=\frac{1}{\lim_{n\to\infty} n^{\frac{1}{n}}} = \frac{1}{1} = 1.$

$z=e^{i\theta}$, $\quad n\cdot e^{in\theta} \not\to 0 \Rightarrow$ divergent at any point $z=e^{i\theta} \in \partial\mathbb{D}$.

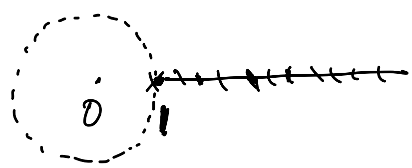
$\sum_{n=1} \frac{z^n}{n^2}$, $\quad R=1$.

$\left|\frac{z^n}{n^2}\right| \le \frac{1}{n^2}$. $\sum \frac{1}{n^2}$ convergent $\Rightarrow \sum_{n=1}\frac{z^n}{n^2}$ convergent at any pt. $z\in\partial\mathbb{D}$.

$f(z)=\sum_{n=1}^{\infty} \frac{z^n}{n^2}$, $\quad f'=\sum_{n=1}^{\infty}\frac{z^{n-1}}{n} = \left(\sum_{n=1}\frac{z^n}{n}\right)\cdot\frac{1}{z} = -\frac{\log(1-z)}{z}$

$\sum_{n=1}^{\infty} \frac{z^n}{n} = g(z)$, $\quad g'(z)=\sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1-z}$.

$\Rightarrow g(z) = -\log(1-z)$.



$N,M\gg1$

$R = \frac{1}{\lim\left(\frac{1}{n}\right)^{\frac{1}{n}}} = \frac{1}{1} = 1.$ $\qquad \underset{\vee}{\underline{\left(\frac{1}{N}+\frac{1}{M}+\sum_{n=M}^{N}\frac{1}{n(n+1)}\right)}}\frac{2}{\delta}<\varepsilon$

$z=e^{i\theta}\neq1$ $\quad \left|\sum_{n=M}^{N}\frac{z^n}{n}\right| = \left|\frac{1}{N}\cdot B_N - \frac{1}{M}B_M - \sum_{n=M}^{N-1}\underbrace{\left(\frac{1}{n+1}-\frac{1}{n}\right)}_{=\,\frac{-1}{n(n+1)}} B_n\right|$

$B_N=\sum_{k=1}^{N} e^{ik\theta} = \frac{e^{i\theta}-e^{i(N+1)\theta}}{1-e^{i\theta}} = e^{i\theta}\frac{1-e^{iN\theta}}{1-e^{i\theta}}$

$|B_N| = \frac{|1-e^{iN\theta}|}{|1-e^{i\theta}|} \le \frac{1}{\delta}\cdot(1+1) = \frac{2}{\delta}$

$\Rightarrow \sum_{n=1}^{\infty}\frac{e^{in\theta}}{n}$ convergent if $e^{i\theta}\neq1$.

$\overset{\theta=0}{e^{i\theta}=1}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n} \sim \int_1^{\infty} \frac{dx}{x} = +\infty.$

- Harmonic functions. $u: \Omega \to \mathbb{R}$ is harmonic if

$$0 = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}}$$

$$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

- f is holomorphic $\Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4}(\Delta u + i \Delta v)$$

$\Rightarrow \quad u = \mathrm{Re}(f), \; v = \mathrm{Im}(f)$ harmonic.

- If Ω is simply connected, and $u: \Omega \to \mathbb{R}$ harmonic

$\Rightarrow \exists$ (conjugate) harmonic fct. v s.t. $f = u + iv$ is holomorphic

Pf: Set $g = 2 \cdot \frac{\partial u}{\partial z} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) u = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$

$$\frac{\partial g}{\partial \bar{z}} = 2 \cdot \frac{\partial^2 u}{\partial \bar{z} \partial z} = \frac{1}{2} \Delta u = 0 \Rightarrow g \text{ is holomorphic.}$$

Because Ω is simply connected, there exists a primitive for g which is by definition a holomorphic fct f s.t. $f' = g$

$f = u_1 + i v_1$ $\qquad \frac{\partial \bar{f}}{\partial z} = \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = 0.$

$$f' = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial \bar{f}}{\partial z} = \frac{\partial}{\partial z}(f + \bar{f}) = \frac{\partial}{\partial z}(2\underbrace{\mathrm{Re}(f)}_{=2u_1}))$$

$$\parallel$$

$$2 \cdot \frac{\partial u_1}{\partial z} = g = 2 \cdot \frac{\partial u}{\partial z} \Rightarrow 2\frac{\partial}{\partial z}(u_1 - u) = 0.$$

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u_1 - u)$$

CR

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x}(u_1 - u) = 0 \\ \frac{\partial}{\partial y}(u_1 - u) = 0 \end{cases} \Rightarrow u_1 - u = \text{const.} \Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y} \\ \frac{\partial u}{\partial y} = \frac{\partial u_1}{\partial y} = -\frac{\partial v_1}{\partial x}. \end{cases}$$

- u harmonic in a disk D_R.

$\Rightarrow \exists f$ holomorphic fct. $f = u + iv$

$$f(z) = \frac{1}{2\pi i}\int_{|z|=R} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

$$\Rightarrow \underline{f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})\, \mathrm{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right) d\varphi.}$$

$\zeta = Re^{i\varphi}$

$$\frac{1}{2}\left(\frac{\zeta + z}{\zeta - z} + \frac{\bar{\zeta} + \bar{z}}{\bar{\zeta} - \bar{z}}\right) = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}$$

$$\Rightarrow u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta}) \cdot \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}\, d\theta \qquad \boxed{\text{Poisson Integral Formula}}$$

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R} f(\zeta)\cdot\frac{1}{2}\left(\frac{\zeta+z}{\zeta-z}+\frac{\bar\zeta+\bar z}{\bar\zeta-\bar z}\right)\frac{d\zeta}{\zeta} \qquad \left(\frac{d\zeta}{\zeta}= i\,d\varphi\right)$$

$$\frac{d\zeta}{\zeta} = i\, d\log\zeta$$

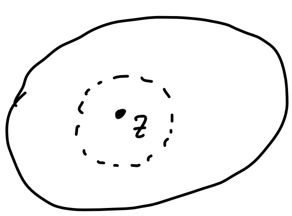
$$\int_{|\zeta|=R} f(\zeta)\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta} = \int f(\zeta)\left(\frac{-\zeta+z}{\zeta-z}+\frac{2\zeta}{\zeta-z}\right)\cdot\frac{1}{\zeta}\,d\zeta$$

$$= \int_{|\zeta|=R}\left(-\frac{f(\zeta)}{\zeta}+2\cdot\frac{f(\zeta)}{\zeta-z}\right)d\zeta = \left(-f(0)+2\cdot f(z)\right)\cdot 2\pi i.$$

$$\int_{|\zeta|=R} f(\zeta)\cdot\left(\frac{\bar\zeta-\bar z+2\bar z}{\bar\zeta-\bar z}\right)\frac{d\zeta}{\zeta} = \underbrace{\int_{|\zeta|=R} f(\zeta)\cdot\frac{d\zeta}{\zeta}}_{= f(0)} + 2\cdot\underbrace{\int_{|\zeta|=R} f(\zeta)\cdot\frac{\bar z}{|\zeta|^2-\bar z\zeta}\,d\zeta}_{=0}$$

- harmonic fct. satisfies maximum principle.

u not constant. u obtains maximum at $z_0 \in \Omega$.



In $B_R(z_0) \subset \Omega$, $u = \operatorname{Re} f$.

$f(B_R(z_0))$ open

$\Rightarrow$ $u(z)$ can not obtain maximum at z_0 contradiction.

- Cauchy Inequality. $f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z} d\zeta$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

$$\Rightarrow |f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{2\pi R}{R^{n+1}} \max_{|\zeta - z_0| = R} |f(\zeta)|$$

$\Longrightarrow$ Liouville Thm: $f: \mathbb{C} \to \mathbb{C}$ entire holomorphic & $|f|$ bounded $\Rightarrow f$ is constant.

If: $f: \mathbb{C} \to \mathbb{C}$ satisfies $\mathrm{Re}(f)$ bounded, then

$$|e^f| = |e^{\mathrm{Re} f} \cdot e^{i \mathrm{Im} f}| = e^{\mathrm{Re} f} \text{ is bounded}$$

$\Rightarrow e^f$ constant $\Rightarrow$ $e^f \cdot f' = 0 \Rightarrow f' \equiv 0 \Rightarrow f = \text{constant}$.

- Little Picard Thm: $f: \mathbb{C} \to \mathbb{C}$ entire fct. Then $f(\mathbb{C})$ can omit at most one value. $\Uparrow$ $F(z) = f(\frac{1}{z})$.

Big Picard: near essential singularity, f can admits at most one value.