$f: U \to V$ conformal (biholomorphic) if $f$ is holomorphic and bijective.

$\Downarrow$

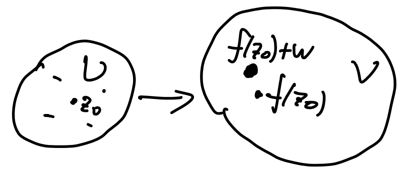
preserves angles.

Thm: If $f$ is holomorphic and (locally) injective, then $f'(z) \neq 0$ for $z \in U$.

Pf: Proof by contradiction. Suppose $f'(z_0) = 0$. Then

$$f(z) - f(z_0) = \underbrace{a_1}_{f'(z_0)=0}(z-z_0) + a_2(z-z_0)^2 + \cdots = a(z-z_0)^k + \underbrace{G(z)}_{(z-z_0)^{k+1}\cdot g(z)}. \qquad (k \geq 2)$$

$$0 = f(z) - (f(z_0) + w) = a\cdot(z-z_0)^k + G(z) - w$$




$$= \underbrace{(a(z-z_0)^k - w)}_{F(z)} + G(z).$$

If $w \neq 0$, always has $k$ roots different.

$$e^{\frac{2\pi i}{k}} \cdot \left(\frac{w}{a}\right)^{\frac{1}{k}} + z_0$$

when $|z - z_0| = \delta$ small. $|w| < \frac{\delta^k \cdot |a|}{2}$

$|G(z)| \leq |z-z_0|^{k+1} \cdot |g(z)| \leq C \cdot |z-z_0|^{k+1} = C \cdot \delta^{k+1}$.

$|F(z)| = |a(z-z_0)^k - w| \geq |a| \cdot |z-z_0|^k - |w|$.

$\geq \frac{|a|}{2} \delta^k$

Rouché $\Rightarrow$ For $|w| < \frac{\delta^k \cdot |a|}{2}$, $F + G$ has the same number of zeros as $F = k \geq 2$.

contradiction $\Rightarrow f'(z_0) \neq 0$. ∎

Rmk: $f'(z_0) \neq 0 \Leftrightarrow f$ is locally injective near $z_0$

$\exists$ nbhd. $V$ of $z_0$ s.t. $f|_V : V \to \mathbb{C}$ is injective.

**Rmk:** $f: U \to \mathbb{C}$ <u>holomorphic and injective</u>

$V = f(U)$ is open (by open mapping thm).

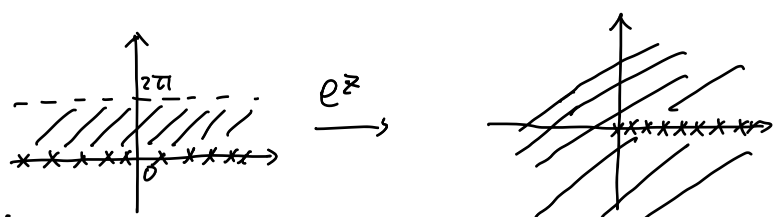
$\Rightarrow f: U \to V$ is biholomorphic (a conformal map).

**Cor:** $f: U \to V$ <u>biholomorphic</u> then $f^{-1}: V \to U$ is holomorphic.

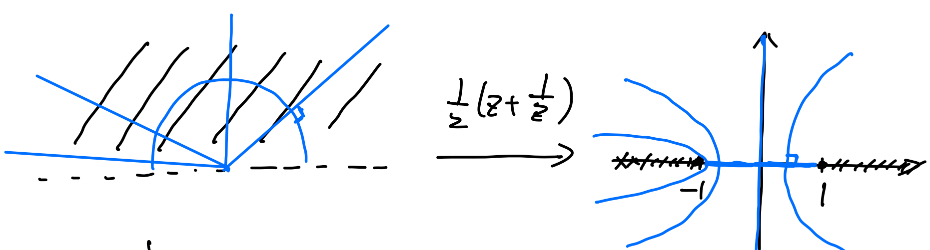
$\Rightarrow$ injective $\Rightarrow f'(z_0) \neq 0$.

$$\lim_{w \to w_0} \frac{\overset{z}{\overset{\|}{f^{-1}(w)}} - \overset{z_0}{\overset{\|}{f^{-1}(w_0)}}}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}.$$

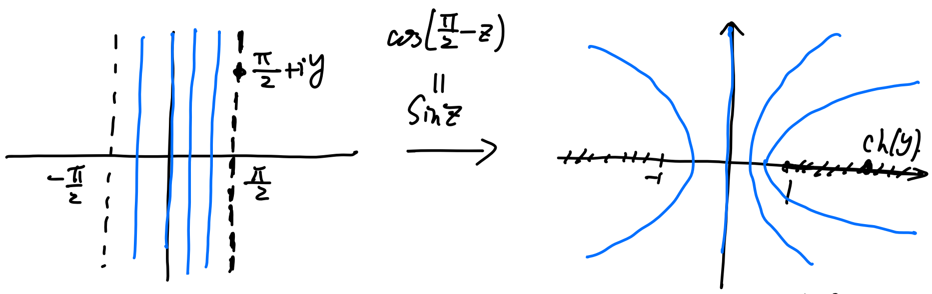
**Ex:**




$\{0 < \mathrm{Im}\, z < 2\pi\}$ $\qquad$ $\mathbb{C} \setminus [0, +\infty)$




$\{\mathrm{Im}\, z > 0\}$ $\qquad$ $\mathbb{C} \setminus (-\infty, -1] \cup [1, +\infty)$




$$\sin\left(\frac{\pi}{2} + iy\right) = \cos(iy) = \frac{e^{i \cdot iy} + e^{-i(iy)}}{2} = \frac{e^{y} + e^{-y}}{2} = ch(y) \underset{1}{\geq}$$

Ex: linear fractional transformation

$$f(z)=\frac{az+b}{cz+d} \qquad a,b,c,d\in\mathbb{C}. \qquad ad-bc\neq 0.$$

$$\| $$

$$\frac{a(z+\frac{d}{c})+b-\frac{ad}{c}}{cz+d}=\frac{a}{c}+\frac{bc-ad}{c^2(z+\frac{d}{c})}.$$

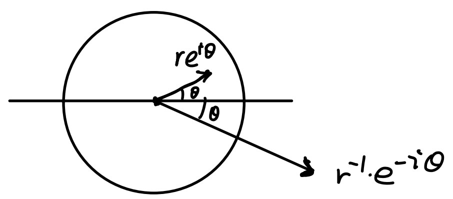
translation: $z\mapsto z+a$

scaling: $z\mapsto c\cdot z$

Inversion: $z\mapsto \frac{1}{z}$

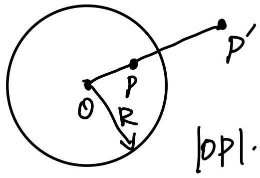
$r\cdot e^{i\theta}\mapsto r^{-1}\cdot e^{-i\theta}$

$$\begin{tikzcd}
z \arrow[r, "\text{trans.}"] & z+\frac{d}{c} \arrow[r, mapsto, "\text{Inv.}"] & \frac{1}{z+\frac{d}{c}} \arrow[d, "\text{scaling}"] \\
\frac{a}{c}+\frac{bc-ad}{c^2(z+\frac{d}{c})} & & \frac{bc-ad}{c^2}\cdot\frac{1}{z+\frac{d}{c}} \arrow[ll, "\text{translation}"']
\end{tikzcd}$$

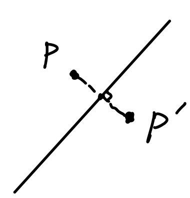



Fact: • linear fractional transformation maps circles to circles
including lines
$\|$
circles with $\infty$ radius.

• linear fractional transformation maps two symmetric pts to two symmetric pts.
(under reflection)

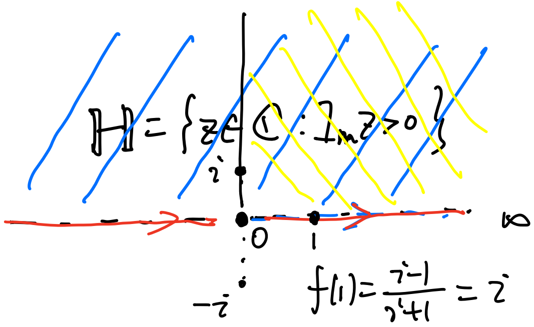



$|OP|\cdot|OP'|=R^2$

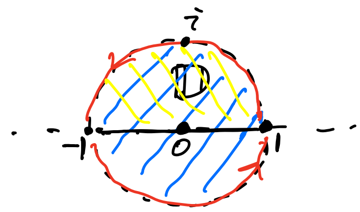


- Any region on one side of a circle can be mapped to a region on one side of any other circle.

$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$




$f(1) = \frac{i-1}{i+1} = i$




$i \mapsto 0$ $\quad \rightsquigarrow \quad \frac{i-z}{i+z} = f(z) : \mathbb{H} \to \mathbb{D}$.

$-i \mapsto \infty$

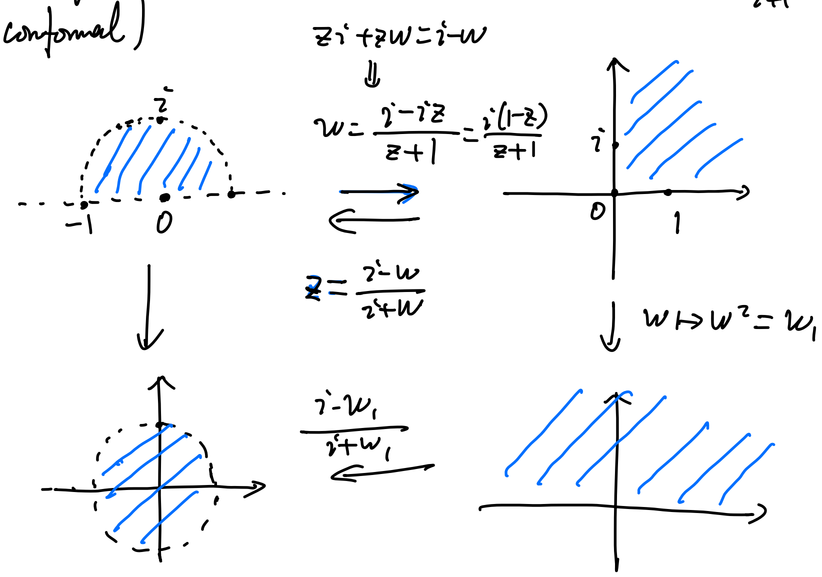
$|f(x)| = \left| \frac{i-x}{i+x} \right| = 1 \quad , \quad f(\infty) = -1$

$y > 0 \quad f(iy) = \frac{i - iy}{i + iy} = \frac{1-y}{1+y} \in (-1, 1)$.

<u>Thm</u> (Riemann mapping theorem) Any simply connected open domain in $\mathbb{C}$ is biholomorphic to the unit disk.
(conformal)

$i \cdot \frac{1-i}{i+1} = i(-i) = 1$

<u>Ex</u>:

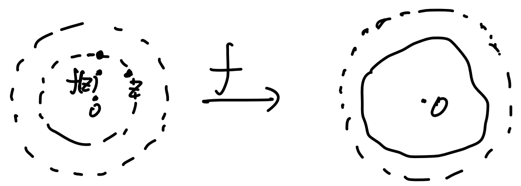


**Schwarz Lemma:** $f: \mathbb{D} \to \mathbb{D}$ holomorphic $f(0)=0$. Then

(i) $|f(z)| \le |z|$, for all $z \in \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$

(ii) If for some $z_0 \neq 0$. $|f(z_0)| = |z_0|$, then $f$ is a rotation $\left(f(z) = e^{i\theta_0} \cdot z\right)$

(iii) $|f'(0)| \le 1$ and if equality holds, then $f$ is a rotation.



Pf: Consider $\boxed{g(z) = \frac{f(z)}{z}}$. holomorphic on $\mathbb{D}$.

$$\lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0) \Rightarrow 0 \text{ is removable sing. for } g.$$

$$\left( \frac{f(z)}{z} = \frac{\overset{0}{f(0)} + f'(0) \cdot z + z^2 \cdot g}{z} = f'(0) + z \cdot g(z) \quad \text{hol. near } 0 \right)$$

Maximum Principle $\underset{(r<1)}{\Rightarrow}$ $\max\limits_{|z| \le r} \left| \frac{f(z)}{z} \right| \le \max\limits_{|z| = r} \frac{|f(z)|}{|z|} \le \frac{1}{r}$

$\Rightarrow |f(z)| \le \frac{|z|}{r}$ for $|z| \le r$. true for any $0 < r < 1$.

For fixed $\underset{\mathbb{D}}{\underset{\Uparrow}{z}}$, let $r \to 1$ to get $|f(z)| \le |z|$. $\Rightarrow \underbrace{\left| \frac{f(z)}{z} \right|}_{g(z)} \le 1$ on $\mathbb{D}$.

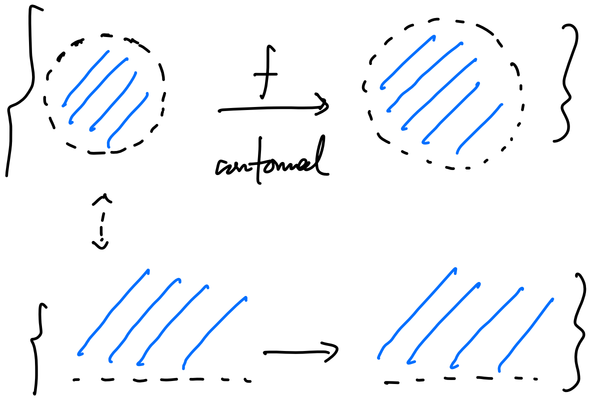
- If for $\overset{\neq 0}{z_0} \in \mathbb{D}$, $|f(z_0)| = |z_0| \Leftrightarrow \left| \frac{f(z_0)}{z_0} \right| = 1 \Rightarrow \frac{f(z)}{z} = c$ constant

  $\Rightarrow f(z) = c \cdot z$. $|f(z_0)| = |c| \cdot |z_0| = |z_0| \Rightarrow |c| = 1 \Rightarrow c = e^{i\theta}$.

  $\Rightarrow f(z) = e^{i\theta} \cdot z$ rotation.

- $|f'(0)| = |g(0)| = 1 \Rightarrow g(z) = \text{constant} \Leftrightarrow f(z) = c\,z \Rightarrow |f'(0)| = |c| = 1 \Rightarrow c = e^{i\theta}$

  $\Rightarrow f(z) = e^{i\theta} \cdot z$ a rotation. ∎




$= \mathrm{Aut}(\mathbb{D})$.

$\supset \{\text{linear fractional transformations}\}$

$= \mathrm{Aut}(\mathbb{H})$.