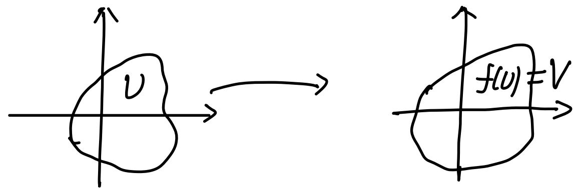
biholomorphic

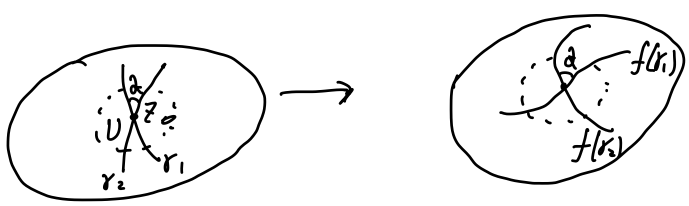
= Conformal map : $f: U^{\subset \mathbb{C}} \to V^{\subset \mathbb{C}}$ is a conformal map if f is holomorphic and bijective (injective & surjective)

$f: U \to \mathbb{C}$
holomorphic



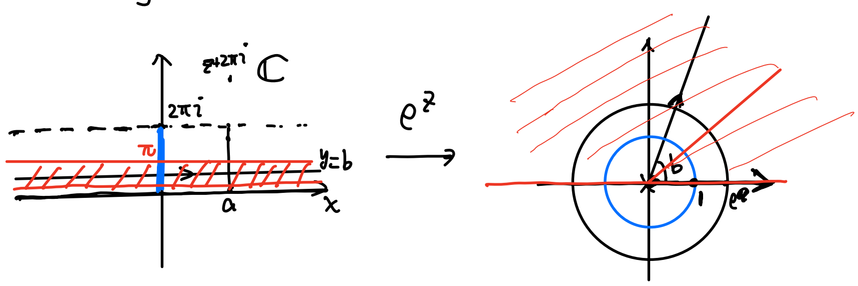
Prop: If f is a conformal map from U to $V = f(U)$, then $f^{-1}: V \to U$ is also a conformal map.

- holomorphic fct. f. Assume $f'(z_0) \neq 0$. Then f is locally injective near z_0.



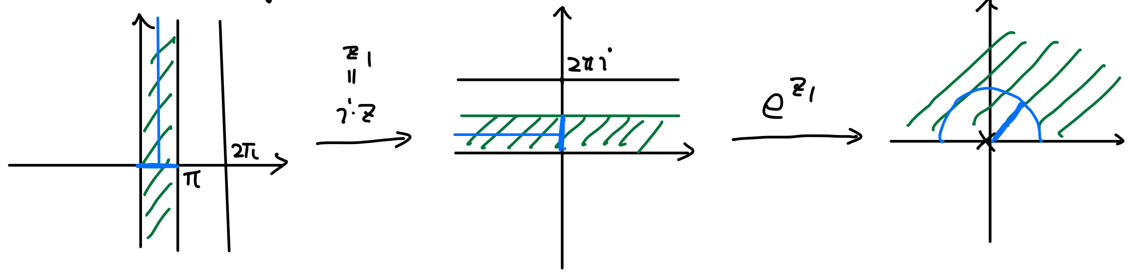
- Ex: $f(z) = e^z = e^{x+iy} = \boxed{e^x e^{iy}}$. $\quad e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z$

$Im(e^z|_{\mathbb{C}}) = \mathbb{C} \setminus \{0\}$.

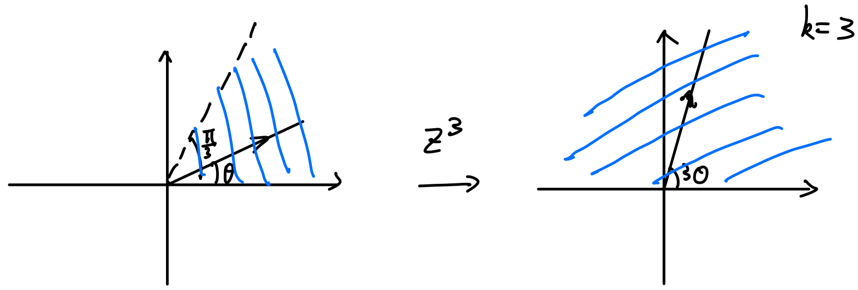


$U = \{z : 0 < Im\, z < \pi\} \xrightarrow{e^z} V = \{w \in \mathbb{C} : Im\, w > 0\}$ conformal map

- $e^{iz} = f(z)$ $f(z+2\pi) = f(z)$ $e^{iz} = e^{i(x+iy)} = e^{-y} e^{ix}$



- $z^k = (r\cdot e^{i\theta})^k = r^k e^{ik\theta}$ k integer.



$$U = \{ z \underset{0}{\neq} : 0 < \arg(z) < \tfrac{\pi}{3} \} \xrightarrow{z^3} V = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$$

- $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z} = f\left(\frac{1}{z}\right)$

$$f(z_1) = f(z_2) \iff \frac{z_1^2+1}{2z_1} = \frac{z_2^2+1}{2z_2} \iff z_1^2 z_2 + z_2 - (z_2^2 z_1 + z_1) = 0$$

$$0 = z_1 z_2 \cdot (z_1 - z_2) - (z_1 - z_2) = (z_1 z_2 - 1)(z_1 - z_2)$$

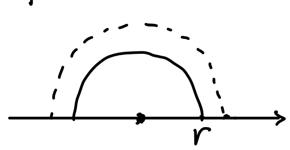
$\Updownarrow$

$z_1 = z_2$ or $\boxed{z_1 z_2 = 1}$

$$U = \{ z \in \mathbb{C},\ |z| < 1,\ \operatorname{Im} z > 0 \}$$

$$\frac{x^2}{\left(\frac{1}{2}(r+r^{-1})\right)^2} + \frac{y^2}{\left(\frac{1}{2}(r-r^{-1})\right)^2} = 1$$

$$\boxed{\begin{cases} x = \frac{1}{2}(r+r^{-1})\cos\theta \\ y = \frac{1}{2}(r-r^{-1})\sin\theta \\ 0 < \theta < \pi \end{cases}}$$



$\longrightarrow$

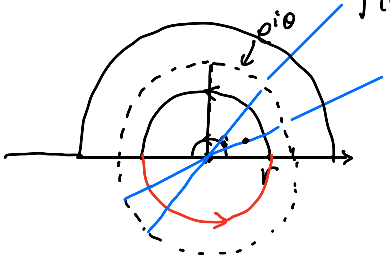
$$z = re^{i\theta},\ 0 < \theta < \pi \mapsto \frac{1}{2}\left(r\cdot e^{i\theta} + r^{-1} e^{-i\theta}\right) = \frac{1}{2}(r + r^{-1})\cos\theta + i\cdot\frac{1}{2}(r - r^{-1})\sin\theta$$

$$\| $$

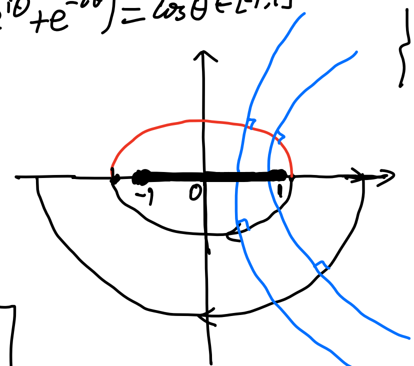
$$\left(r\cdot\cos\theta + ir\sin\theta + r^{-1}(\cos\theta - i\cdot\sin\theta)\right)$$

$$f(e^{i\theta}) = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta}) = \cos\theta \in [-1,1]$$

$$\begin{cases} x = \tfrac{1}{2}(r+r^{-1})\cos\theta \\ y = \tfrac{1}{2}(r^{-1}-r)\cdot\sin(-\theta) \end{cases}$$



$\longrightarrow$



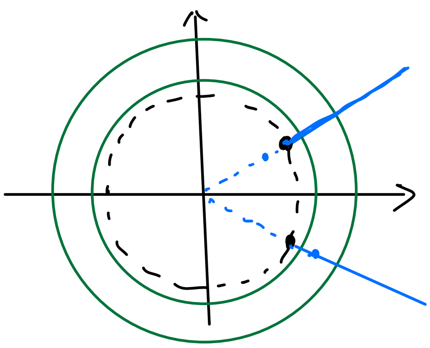
$$\theta = \tfrac{\pi}{2}: \begin{cases} x = 0 \\ y = \tfrac{1}{2}(r^{-1}-r)\cdot(-1) \end{cases}$$

For fixed θ, $\boxed{\dfrac{x^2}{\cos^2\theta} - \dfrac{y^2}{\sin(-\theta)^2} = 1}$

$$D^{\times} = D \setminus \{0\} \xrightarrow[\text{Conformal}]{\frac{1}{2}(z+\frac{1}{z})} \mathbb{C} \setminus [-1,1]$$

$D = \{z \in \mathbb{C} : |z| < 1\}$

$$\mathbb{C} \setminus \overline{D} = \{z \in \mathbb{C} : |z| > 1\} \xrightarrow{\text{conformal}} \mathbb{C} \setminus [-1,1]$$



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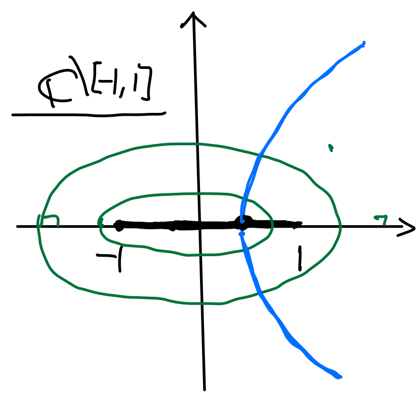
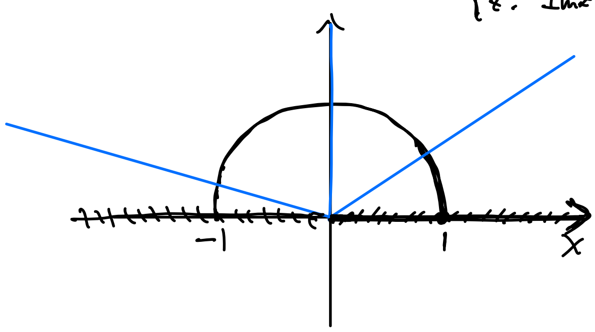


$$\{z : \operatorname{Im} z > 0\} \xrightarrow{\frac{1}{2}(z+\frac{1}{z})} \mathbb{C} \setminus (-\infty,-1] \cup [1,\infty).$$



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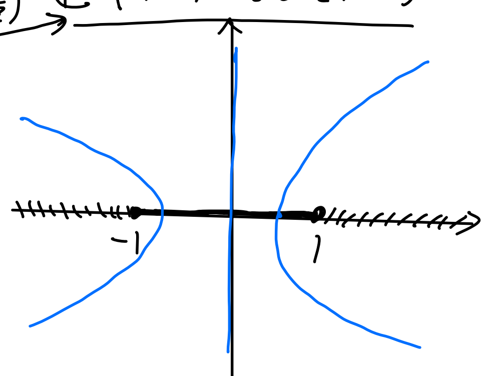
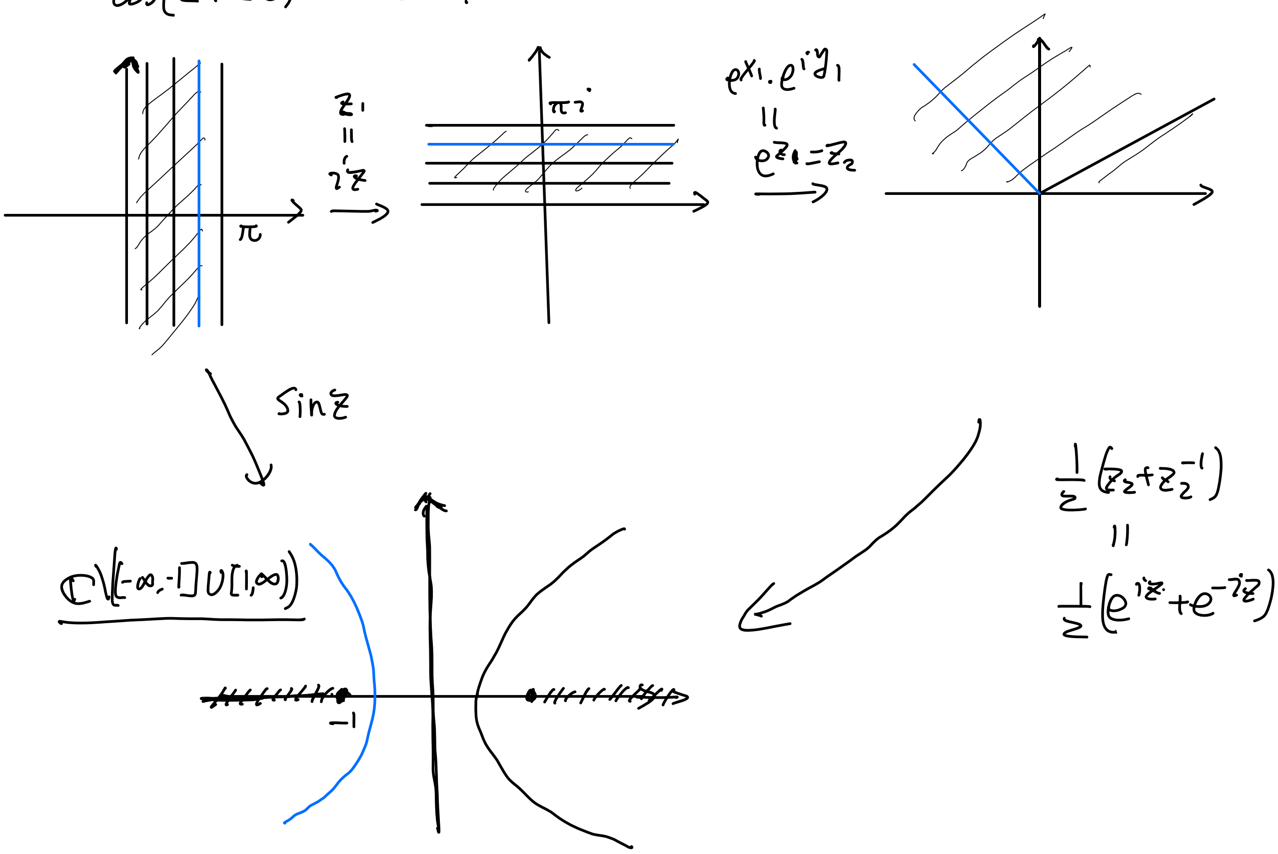


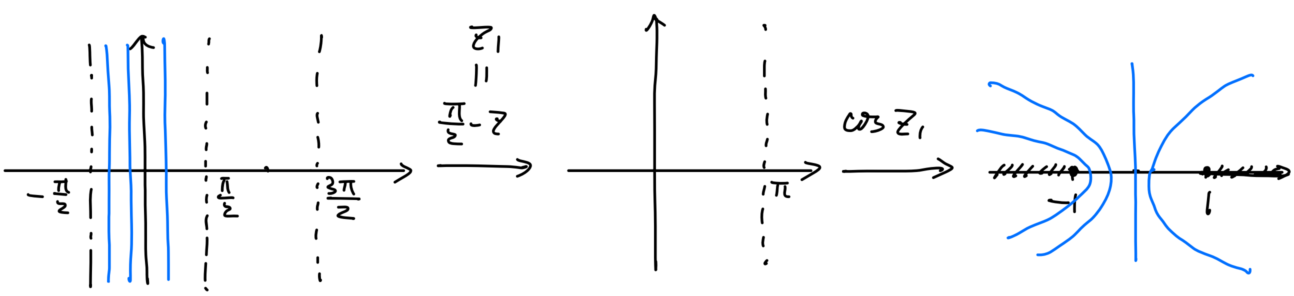
$x \mapsto \tfrac{1}{2}(x+x^{-1}) \geq 2$

Ex: $f(z) = \cos z = \dfrac{e^{iz}+e^{-iz}}{2}$ $\quad\quad \sin(z) = \cos\left(\frac{\pi}{2} - z\right)$

$\cos(z+2\pi) = \cos(z).$ $\quad \cos(z+\pi) = -\cos z$



$\sin(z) = \cos\left(\frac{\pi}{2} - z\right)$



$\sin(\pi+z) = -\sin z.$