Thm: $\underline{\Omega \text{ simply connected}}$. $f(z)$ holomorphic $\neq 0$ in Ω anywhere

Then there exists holomorphic function $g(z)$ in Ω s.t. $e^{g(z)} = f(z)$

i.e. can define $\log f$ $= g(z)$.

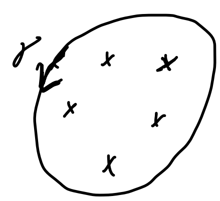
In particular, if $0 \notin \Omega$, then there exists a branch of $\log z$ in Ω.

Pf: Define $\underline{\displaystyle\int_{z_0}^{z} \frac{f'}{f}\, dz = g(z)}$.

- Ω simply connected $\Rightarrow$ independent of path from $z_0 \to z$
- $\underline{f \neq 0.\ \text{globally defined}}$.

- $\gamma \subset \Omega$ closed curve. γ encloses a simply connected dom. D.

f is meromorphic in Ω. f has no zeros or poles on γ
($\exists$ zeros and poles)



$$\int_\gamma \left(\frac{f'}{f}\right) dz$$

$(\operatorname{ord}_{z_0} f = n)$

- z_0 is a zero: $\underline{f(z) = (z-z_0)^n \cdot g(z)}$ near z_0. $g(z)$ hol. near z_0, $g(z_0) \neq 0$

$f'(z) = n \cdot (z-z_0)^{n-1} \cdot g + (z-z_0)^n \cdot g'$

$$\frac{f'}{f} = \frac{n \cdot (z-z_0)^{n-1} \cdot g + (z-z_0)^n \cdot g'}{(z-z_0)^n \cdot g} = \frac{n}{z-z_0} + \frac{g'}{g}$$ hol. near z_0

$\Rightarrow$ z_0 is a pole for $\frac{f'}{f}$ (of order 1), $\operatorname{res}_{z_0} \frac{f'}{f} = n$.

z_0 is a pole: $f(z) = (z-z_0)^{-n} \cdot g(z)$. $\underline{(\operatorname{ord}_{z_0} f = -n)}$

$\Rightarrow$ $\frac{f'}{f} = \frac{-n}{z-z_0} + \frac{g'}{g}$ $\Rightarrow$ z_0 is a pole for $\frac{f'}{f}$ $\operatorname{res}_{z_0} \frac{f'}{f} = -n$.

$$\underline{\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_i \mathrm{res}_{z_i} f} = \sum_{\substack{z_i \text{ zero} \\ \text{or pole}}} \mathrm{ord}_{z_i}(f) = \{\#\text{ zeros of } f \text{ inside } \gamma\} - \{\#\text{ poles of } f \text{ inside } \gamma\}$$

(Argument Principle)

$$\frac{f'}{f} = (\underbrace{\log f}_{\uparrow})' = (\underline{\mathrm{Log}|f| + i\cdot \arg(f)})'$$

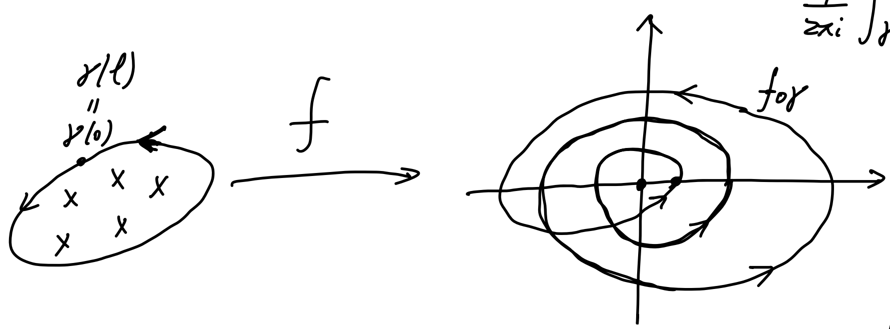
locally defined away from zeros and poles

$$\int_\gamma \frac{f'}{f}\,dz = \int_\gamma d\,\mathrm{Log}|f| + i\, d\arg(f) = i\cdot\left(\arg(f(\gamma(\ell))) - \arg(f(\gamma(0)))\right)$$

$$\int_\gamma \frac{f'}{f}\,dz = i\cdot 2\pi\cdot(\text{winding number})$$

$$\int_\gamma d\,\mathrm{Log}|f| = \mathrm{Log}|f|(\gamma(\ell)) - \mathrm{Log}|f|(\gamma(0)) = 0$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \frac{i6\pi}{2\pi i} = 3.$$



Thm (Rouché's Thm) f and g holomorphic in Ω. γ closed curve enclosing simply connected domain (eg. circles)

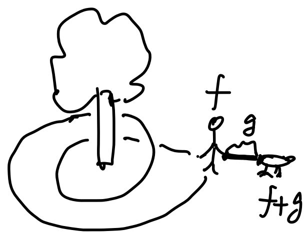
$\boxed{|f(z)| > |g(z)| \text{ for all } z \in \gamma}$

Then f and $f+g$ have the same number of zeros inside γ.

Pf: $\underline{f_t = f + tg}$, $0 \le t \le 1$. $f_0 = f$, $f_1 = f+g$.

$$n_t = \frac{1}{2\pi i}\int_\gamma \frac{f_t'}{f_t}\,dz = \#\text{ zeros of } f_t \text{ inside } \gamma \in \mathbb{Z}_{\ge 0}.$$

- $|f_t| = |f+tg| \geq |f| - t|g| > 0$
- n_t is continuous in $t \in [0,1] \implies \overset{\overset{n_1}{\shortparallel}}{n_t} \equiv n_0 \in \mathbb{Z}_{\geq 0}.$ ■



Ex: $\underbrace{z^4 - 6z + 3}_{\overset{\shortparallel}{P(z)}} = 0$ 1. # of roots inside $|z| < 1$.

2. # of roots inside annulus $1 < |z| < 2$

- $\gamma_1 : |z| = 1$. $|\underbrace{z^4+3}_{\overset{\shortparallel}{g}}| \leq |z|^4 + 3 = 4 < |\underbrace{-6z}_{\overset{\shortparallel}{f}}| = 6$

$\implies$ # of roots for $\underset{\underset{f+g}{\shortparallel}}{P(z)}$ = # of roots $(-6z)$ inside $|z|=1$ $= 1$.

- $\gamma_2 : |z| = 2$. $|\underset{\underset{g}{\shortparallel}}{-6z+3}| \leq 6 \cdot |z| + 3 = 6 \cdot 2 + 3 = 15 < 16 = |\underset{\underset{f}{\shortparallel}}{z^4}|$

$\implies$ # of roots for $P(z)$ inside $(|z|=2)$ = # roots for f $= 4$

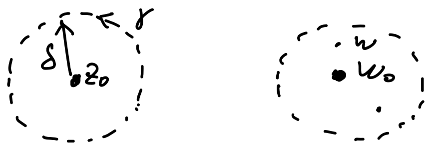
$\implies$ $P(z)$ has $\overset{\overset{4-1}{\shortparallel}}{3}$ roots inside the annulus $1 < |z| < 2$.

Thm (Open mapping theorem) If f is holomorphic and nonconstant in a region Ω, then f is open i.e. f maps open sets to open sets.

Pf: $z_0 \in U$. $w_0 = f(z_0) \in f(U)$ $f(z) - w$ has a zero in U

Prove all w near w_0, $w \in f(U)$ i.e. $w = f(z)$ for some $z \in U$.

$$\boxed{f(z) - w_0 = f(z) - f(z_0) = (z-z_0)^n \cdot g(z). \quad g(z_0) \neq 0}$$



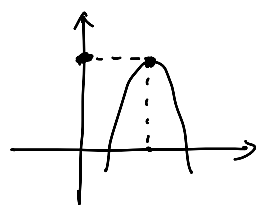
$$f(z) - w = \underbrace{(f(z) - w_0)}_{F} + \underbrace{(w_0 - w)}_{G}.$$

$\left(\exists \delta > 0.\ \text{s.t.}\ |f(z) - w_0| \geq \varepsilon > 0 \text{ on } C(z_0, \delta) \subset U \right)$

$\underbrace{|f(z) - w_0|}_{F} \geq \varepsilon > \underbrace{|w_0 - w|}_{G}$. on $C(z_0, \delta)$ $\Rightarrow$ # zeros for $f(z) - w = F + G$ $=$ # zeros for $f(z) - w_0 = F \geq 1$.

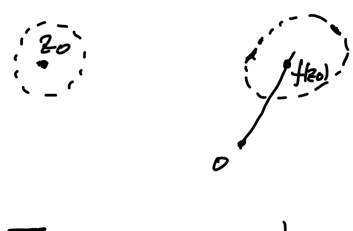
$(f(z_0) - w_0 = 0)$ ∎

real differentiable case



Thm (Maximum modulus principle) If f is not constant, holomorphic in Ω, then $|f|$ cannot obtain a (local) maximum in Ω.

Pf: Suppose $|f(z_0)| = \max_{z \in \Omega} |f(z)|$ $z_0 \in \Omega^\circ$.



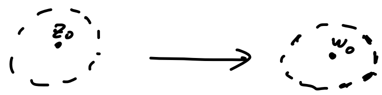
f is open mapping $\Downarrow$ $\exists \underset{f(\Omega)}{\overset{f(z)}{w}}$ near $f(z_0)$ s.t. $|w| > |f(z_0)|$

contradiction ∎

Cor: Assume that $\overline{\Omega}$ is compact. Then $\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \overline{\Omega} \setminus \Omega} |f(z)| = \sup_{\overline{\Omega}} |f(z)|$.

f is holomorphic in Ω and continuous on $\overline{\Omega}$

$$(z-z_0)^n \cdot g(z) = f(z) - w_0$$



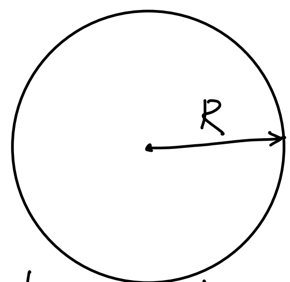
$$f(z) - w$$

$$z \to z^2$$

f differentiable, $f'(x_0) \neq 0$.

Ex: $$P(z) = \underbrace{z^n}_{f} + \underbrace{a_{n-1} \cdot z^{n-1} + \dots + a_1 z + a_0}_{g}$$

$$\frac{|g|}{|z^n|} \le \frac{|a_{n-1}| \cdot |z|^{n-1} + \dots + |a_1||z| + |a_0|}{|z|^n} \xrightarrow{|z| \to +\infty} 0$$



$$\underbrace{|f|}_{= R^n \neq 0} > |g| \Rightarrow \# \text{ zeros of } \underbrace{P}_{= f+g} = \underbrace{\# \text{ zeros of } f = z^n \text{ inside } \{|z| = R\}}_{= n}$$

$\Rightarrow$ $P(z)$ has exactly n roots. (Fundamental Thm of Algebra)