f holomorphic in $\Omega \setminus \{z_0\}$. Three cases

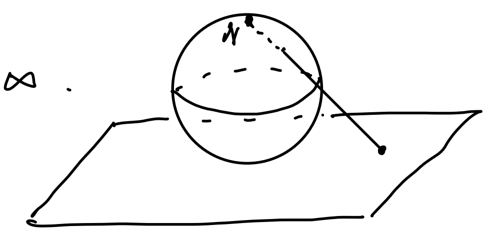
1. z_0 is removable $\Leftrightarrow$ f is locally bounded near z_0
2. z_0 is a pole $\Leftrightarrow$ $|f(z)| \to +\infty$ as $z \to z_0$
3. z_0 is an essential sing. $\Leftrightarrow$ $\mathrm{Im} f|_{D_r(z_0)\setminus\{z_0\}}$ is dense (Casorati-Weierstrass)

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n \cdot (z-z_0)^n$$ Laurent series on $D_r(z_0) \setminus \{z_0\}$.

$$\begin{cases} z_0 \text{ is removable} \Leftrightarrow f(z) = a_0 + a_1(z-z_0) + \cdots \\ \text{pole} \Leftrightarrow f(z) = \dfrac{a_{-n}}{(z-z_0)^n} + \cdots + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots \\ \text{essential} \Leftrightarrow a_{-k} \neq 0 \text{ for infinitely many } k \in \mathbb{Z}_{>0}. \end{cases}$$

∞ is removable / pole / essential for $f: \mathbb{C} \setminus D_r(0) \to \mathbb{C}$ $\overset{\text{def}}{\Longleftrightarrow}$ 0 is removable / pole / essential $\underline{F(z) = f\left(\frac{1}{z}\right)}$

$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong S^2$ $\quad \frac{1}{\infty} = 0, \ \frac{1}{0} = \infty$.



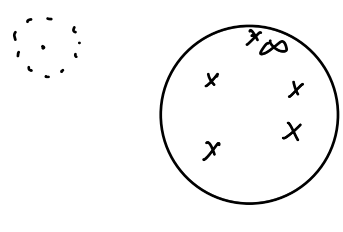
Thm: $\underline{\text{Meromorphic functions on } \overline{\mathbb{C}}}$ $\overset{\text{are}}{\Longleftrightarrow}$ rational functions

(at worst poles)

$f(z) = \dfrac{P(z)}{Q(z)}$ $\quad F(z) = f\left(\frac{1}{z}\right) = \dfrac{\widetilde{P}(z)}{\widetilde{Q}(z)}$

Pf: "$\Rightarrow$" poles are isolated

z is pole for f $\Leftrightarrow$ z is zero for $\frac{1}{f}$



$\Rightarrow$ finitely many poles : $\underline{z_1, \cdots, z_n, \infty}$

$$f(z) = \underbrace{\frac{a_{-m}}{(z-z_k)^m} + \cdots + \frac{a_{-1}}{z-z_k}}_{= f_k = \frac{p_k(z)}{(z-z_k)^m}} + \underline{g_k(z)}. \qquad k=1,\cdots,n$$

$\underline{f_k(z)}$ only has pole at $z=z_k$

∞ is removable

$$\lim_{z\to 0} \left|f_k\left(\tfrac{1}{z}\right)\right| = \lim_{z\to\infty} |f_k(z)| \le \sum_{l=1}^{m} \frac{a_{-l}}{|z-z_k|^l} \to 0$$

$$H = f(z) - \sum_{k=1}^{n} f_k(z) - f_\infty(z)$$

only has pole at $0 \in \overline{\mathbb{C}}$

$$f\left(\tfrac{1}{z}\right) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z) \text{ near } 0$$

$$\Updownarrow$$

$$f(z) = \tilde{f}_\infty\left(\tfrac{1}{z}\right) + \tilde{g}_\infty\left(\tfrac{1}{z}\right) \text{ near } \infty$$

only has pole at $\infty \in \overline{\mathbb{C}}$

$\Rightarrow H$ is holomorphic on $\overline{\mathbb{C}}$ $\Rightarrow H$ is holomorphic on $\mathbb{C}$ and $|f(z)|$ is bounded near ∞.

$\Rightarrow H$ is a bounded entire function.

$\Rightarrow H$ is constant. (Liouville Thm)

$\Rightarrow f(z) = c + \sum_{k=1}^{\infty} f_k + f_\infty$ is rational $\qquad f_\infty(z) = \tilde{f}_\infty\left(\tfrac{1}{z}\right)$ is rational

$\blacksquare$

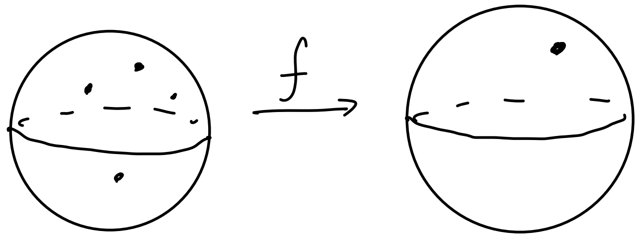
Rmk. · Rational function $\Longleftrightarrow$ holomorphic map to $S^2 = \mathbb{C} \cup \{\infty\}$

· polynomial fct. P $\Rightarrow$ rational function $\deg P = \#\{P^{-1}(w)\}$.

rational function on $\overline{\mathbb{C}}$

$\#\{f^{-1}(w)\}$ does not depend on w

$= \deg f = \max\{P, Q\}$.



$$f(z) = \frac{P(z)}{Q(z)} = w \iff \underline{P(z) = w \cdot Q(z) \iff P(z) - w \cdot Q(z) = 0}$$

(P, Q relatively prime i.e. no common factors)

Example of holomorphic function

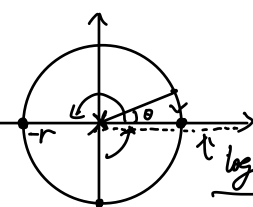
$$P(z),\quad \frac{P(z)}{Q(z)},\quad e^{z},\quad \sin(z),\quad \cos(z),\quad \tan(z) = \frac{\sin(z)}{\cos(z)}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\log(z) = \log(r \cdot e^{i\theta}) = \underline{\log r + i \cdot \theta}$$

$$r \cdot e^{i\theta} = |z| \cdot e^{i\theta} = |z| \cdot e^{i(\theta + 2\pi)}$$

not defined at $z = 0$



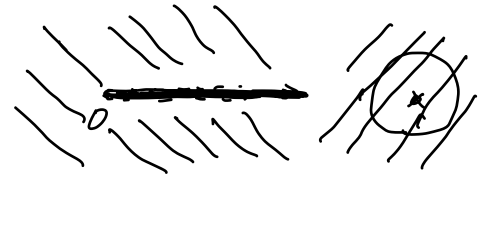
$\log(re^{i\pi}) = \log(-r) = \log r + i\pi$

$\underline{\log r + i\theta \to \log r + i \cdot 2\pi}$

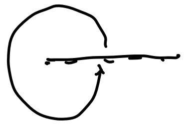
$\log(-ri) = \log r + i \cdot \frac{3\pi}{2}$

<u>Thm</u>: If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists <u>a holomorphic function g on Ω s.t</u>

$$e^{g(z)} = f(z).$$



$$\boxed{\underline{\text{Cor}}:\ f(z)=z,\ \ \Omega=\mathbb{C}\setminus[0,+\infty)\ \text{Simply connected} \Rightarrow e^{g(z)} = z = e^{g(z)+2\pi i\cdot k}\quad k\in\mathbb{Z}.}$$



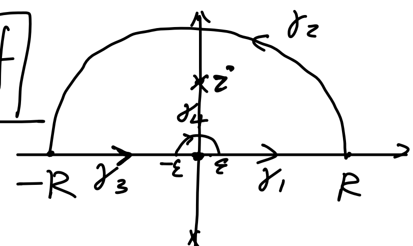
$\log z = \log|z| + i\cdot \arg(z)$

$\underline{0<\arg(z)<2\pi}$. a branch of $\log z$

$\log z = \log|z| + i\arg(z) + \underline{2\pi i k}$, is a branch of $\log z$.

$\underline{\text{Ex}}$: $\displaystyle\int_0^{+\infty} \frac{\log x}{(1+x^2)^2}\,dx \qquad f(z)=\frac{\log z}{(1+z^2)^2}$

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_i \operatorname{res}_{z_i} f = 2\pi i\cdot \operatorname{res}_i f}$$



$(1+z^2)^2=0 \Leftrightarrow z^2+1=0 \Leftrightarrow z=\pm i$

$$f(z)=\frac{\log z}{(z^2+1)^2} = \frac{\log z}{(z+i)^2(z-i)^2} = \frac{g(z)}{(z-i)^2}\quad g(i)\neq 0.$$

pole of order 2. $\underline{\left(\log i = \log 0 + i\cdot\frac{\pi}{2}\neq 0\right)}$

$\frac{d}{dz}\log z = \frac{1}{z}$

$$\operatorname{res}_i f = \left(\frac{d}{dz}(z-i)^2 f(z)\right)\Big|_{z=i} = \frac{d}{dz}\frac{\log z}{(z+i)^2}\Big|_{z=i} = \left(\frac{1}{z}\cdot\frac{1}{(z+i)^2} - \frac{\log z}{(z+i)^3}\cdot 2\right)\Big|_{z=i}$$

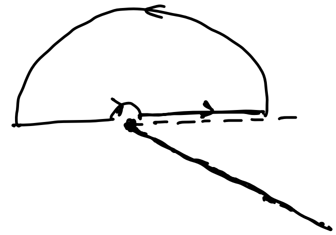
$$\left(\operatorname{res}_i f = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)\Big|_{z=z_0}\right) = \frac{1}{i}\cdot\frac{1}{-4} + \frac{i\cdot\frac{\pi}{2}}{+8i}\cdot 2$$

$$= \frac{i}{4}+\frac{\pi}{8}$$

$(2i)^3 = 8\cdot i^3 = -8i$

$\log z = \log|z| + i\theta$

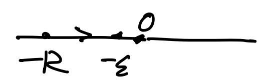
$-\theta_0 < \theta < 2\pi - \theta_0$



$$\int_{\gamma_1} f(z)\,dz = \int_\varepsilon^R \frac{\log x}{(1+x^2)^2}\,dx \xrightarrow[R\to+\infty]{\varepsilon\to 0} \int_0^{+\infty}\frac{\log x}{(1+x^2)^2}\,dx$$

$$\int_{\gamma_2} f(z)\,dz \xrightarrow{R\to+\infty} 0 \xleftarrow{\varepsilon\to 0} \int_{\gamma_4} f(z)\,dz$$

$$\int_{\gamma_3} f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{\log z}{(1+z^2)}\,dz = \int_{-R}^{-\varepsilon} \frac{\log(-x)+i\cdot\pi}{(1+x^2)^2}\,dx$$



$\log z = \log|z| + i\cdot\arg(z)$

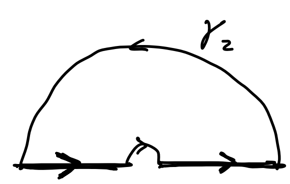
$$= \int_{R}^{\varepsilon} \frac{\log t + i\pi}{(1+t^2)^2}\,d(-t)$$

$$= \int_{\varepsilon}^{R} \frac{\log t + i\pi}{(1+t^2)^2}\,dt \xrightarrow[R\to+\infty]{\varepsilon\to 0} \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx + i\cdot\int_0^\infty \frac{\pi}{(1+t^2)^2}\,dt$$

$$\int_{\gamma_1} + \overset{\overset{0}{\nearrow_{R\to+\infty}}}{\int_{\gamma_2}} + \int_{\gamma_3} + \overset{\overset{0}{\uparrow_{\varepsilon\to 0}}}{\int_{\gamma_4}} = 2\pi i\cdot \mathrm{res}_i f$$

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx \quad \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx + i\cdot\pi\int_0^\infty \frac{dt}{(1+t^2)^2} = 2\pi i\cdot\left(\frac{i}{4}+\frac{\pi}{8}\right)$$

$$2\cdot I + i\cdot\pi\int_0^\infty \frac{dt}{(1+t^2)^2} = -\frac{\pi}{2} + i\cdot\frac{\pi^2}{4}$$



$$\Rightarrow I = -\frac{\pi}{4} = \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx$$

$$\int_{\gamma_4} \frac{\log z}{(1+z^2)^2}\,dz = \int_{\pi}^{0} \frac{\log\varepsilon + i\theta}{\underset{\downarrow \varepsilon\to 0,\ 0}{(1+(\varepsilon\cdot e^{i\theta})^2)^2}}\,\varepsilon\cdot e^{i\theta}\cdot i\,d\theta \xrightarrow{\varepsilon\to 0} 0 \qquad \varepsilon\cdot\log\varepsilon \xrightarrow{\varepsilon\to 0} 0$$

$\gamma_4:\ \theta \mapsto \overset{z}{\overset{\shortparallel}{\varepsilon\cdot e^{i\theta}}},\ \pi\to 0$

$$\left(\frac{\log R}{(1+R^2)^2}\cdot\pi\cdot R \xrightarrow{R\to+\infty} 0\right)$$

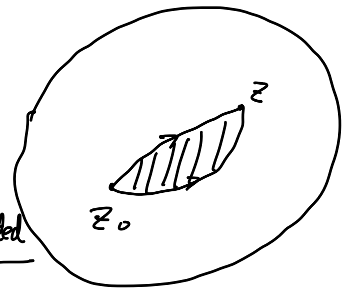
Thm: $f \neq 0$ in $\Omega \leftarrow$ simply connected $\Longrightarrow$ define $\underline{\log f(z) = g}$

$e^g = f$

Pf: $(\underline{\log f})' = \dfrac{f'(z)}{f(z)}$ holomorphic in Ω

$g(z) = \displaystyle\int_{z_0}^{z} \frac{f'(z)}{f(z)}\, dz$ is well defined

because Ω is simply connected



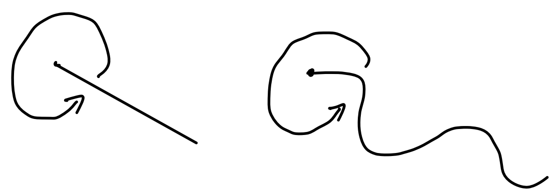
$$\frac{d}{dz}\left(\underline{f(z)\cdot e^{-g}}\right) = f'\cdot e^{-g} - f\cdot e^{-g}\cdot g' = fe^{-g}\left(\frac{f'}{f} - g'\right) = 0.$$

$$g' = \frac{f'}{f}$$

$\Rightarrow f(z)\cdot e^{-g} = f(z_0)\cdot e^{-g_0} = c_0 \neq 0$

$\Rightarrow f(z) = c_0 \cdot e^{g} = e^{g + \log c_0}$

- $\mathrm{Log}\, z = \log|z| + i\cdot \arg(z)$.



- $z^n = z \cdots z$, $\quad z^{-n} = \dfrac{1}{z^n}$.

$\alpha \in \mathbb{C}$, $\quad z^\alpha = e^{\alpha \log z}$ a branch on Ω simply connected, $0 \notin \Omega$

$z^\alpha = (e^{\log z})^\alpha$

$\alpha = z^{\frac{1}{2}}$ on $\mathbb{C} \setminus$ ray.

$z^{\frac{1}{2}} = (r\cdot e^{i\theta})^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\cdot\frac{\theta}{2}}$

$1^{\frac{1}{2}} = \{-1, 1\}$