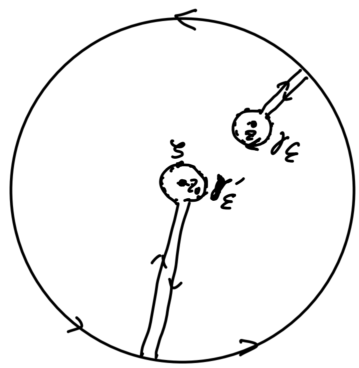
f holomorphic in $\Omega \setminus \{z_0\}$.

3 cases:
- removable singularity ($\Leftrightarrow \overset{|f|}{f}$ is bounded near z_0)
  (define $f(z_0)$ s.t. f becomes holomorphic in Ω.)
- pole singularity ($\Leftrightarrow |f(z)| \to +\infty$ as $z \to z_0$)
- essential singularity.

Thm (Riemann) If f is bounded on $\Omega \setminus \{z_0\}$, then z_0 is removable.

Pf: $g(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$ is holomorphic inside C.

$\neq$
$f(z)$.



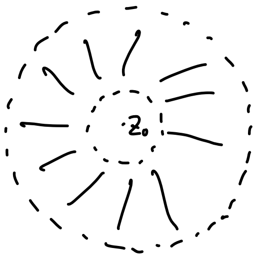
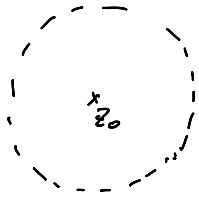
$$0 = \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta + \underbrace{\int_{\gamma_\varepsilon'} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{\downarrow\, \varepsilon \to 0} + \underbrace{\int_{\gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{\parallel\ z + \varepsilon e^{i\theta}}$$

$\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \to 2\pi i \cdot g(z)$

$\int_{\gamma_\varepsilon'} \frac{f(\zeta)}{\zeta - z}\, d\zeta \to 0$

$$\int_{\gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta = -\int_0^{2\pi} \frac{f(z + \varepsilon e^{i\theta})}{\cancel{\varepsilon e^{i\theta}}} \cancel{\varepsilon e^{i\theta}}\, i\, d\theta \xrightarrow{\varepsilon \to 0} -2\pi i \cdot f(z).$$

$$\left| \int_{\gamma_\varepsilon'} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| \leq \int_{\gamma_\varepsilon'} \frac{|f(\zeta)|}{|\zeta - z|}\, |d\zeta| \leq C \cdot 2\pi\varepsilon \xrightarrow{\varepsilon \to 0} 0$$

Fact:  

f holomorphic $r < |z-z_0| < R$.

$\Downarrow$

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \cdot (z-z_0)^n$$
$$= \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n\cdot(z-z_0)^n.$$

Laurent Series

z_0 is removable $\Leftrightarrow$ $a_{-n}=0$, $\forall\ n\geq 1$.

z_0 is a pole $\Leftrightarrow$ $\exists N\in\mathbb{Z}_{>0}$, $a_{-n}=0$. for $n\geq N$

$$\frac{a_{-n}}{(z-z_0)^n}+\frac{a_{-n}}{(z-z_0)^{n-1}}+\cdots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1\cdot(z-z_0)+\cdots$$

z_0 is essential $\Leftrightarrow$ ∞ many terms of negative powers.

Cor: z_0 is a pole $\Leftrightarrow$ $|f(z)|\to+\infty$ as $z\to z_0$.

Pf: · z_0 is a pole $\Leftrightarrow$ z_0 is a zero of $\frac{1}{f}$ $\Rightarrow \lim\limits_{z\to z_0}\frac{1}{f(z)}=0$

$$\Rightarrow \lim_{z\to z_0}|f(z)|=+\infty$$

· $|f(z)|\to+\infty$ ($z\to z_0$) $\Rightarrow$ $\frac{1}{|f(z)|}\xrightarrow{z\to z_0} 0$ $\Rightarrow$ z_0 is removable sing. for $\frac{1}{f(z)}$. $\left(\frac{1}{f}\right)(z_0)=0$

$\frac{1}{|f(z)|} = \left|\frac{1}{f(z)}\right|$

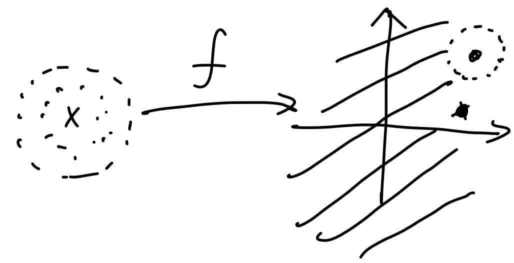
$\Rightarrow z_0$ is a pole for f

Essential: not a removable or a pole.

Thm (Casorati–Weierstrass) z_0 is essential. $z_0 \in D_r(z_0) \subset \Omega$ any small disk

$\Longrightarrow$ the image $f(D_r(z_0)\setminus z_0)$ is dense in $\mathbb{C}$.

Pf: Proof by contradiction.



suppose not dense. $\exists\, w \in \mathbb{C}, \delta > 0$

$|f(z) - w| > \delta$ for $z \in D_r(z_0)\setminus z_0$.

$g(z) = \dfrac{1}{f(z) - w}$ is bounded near $z_0 \Rightarrow z_0$ is a removable sing.

$g(z_0) = 0 \Rightarrow z_0$ is a pole for $f(z) - w$. $\Rightarrow$ not essential.

$g(z_0) \neq 0 \Rightarrow f(z) - w$ is holomorphic near $z_0 \Rightarrow$ not essential.

Contradiction $\blacksquare$

Ex: $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots$ misses $0 \in \mathbb{C}$

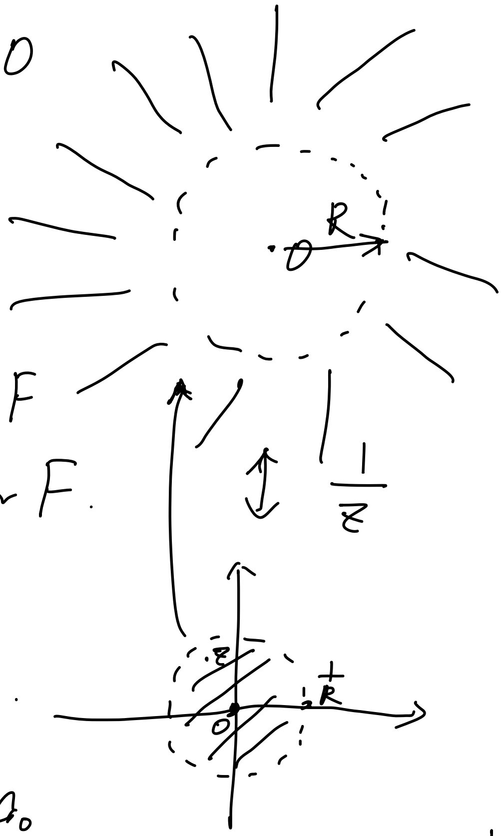
$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots = \dfrac{e^{i\frac{1}{z}} - e^{-i\frac{1}{z}}}{2i} = w$. solvable

Rmk: Big Picard Thm: $f(D_r(z_0)\setminus z_0)$ misses at most 1 pt and takes each value infinitely many times.

$$e^{\frac{1}{z}} = e^{\frac{1}{z} + 2\pi i k} = e^{\frac{1}{\left(\frac{1}{\frac{1}{z} + 2\pi i k}\right)}} \quad \forall\, k \in \mathbb{Z}.$$

• Assume f is holomorphic on $\mathbb{C}\setminus D_R(0)$.

$F(z)=f\left(\frac{1}{z}\right)$ holomorphic near 0 (inside $D_{\frac{1}{R}}(0)$)



∞ is removable for f $\Longleftrightarrow$ 0 is removable for F

pole $\Longleftrightarrow$ 0 is a pole for F

essential $\Longleftrightarrow$ 0 is essential for F.

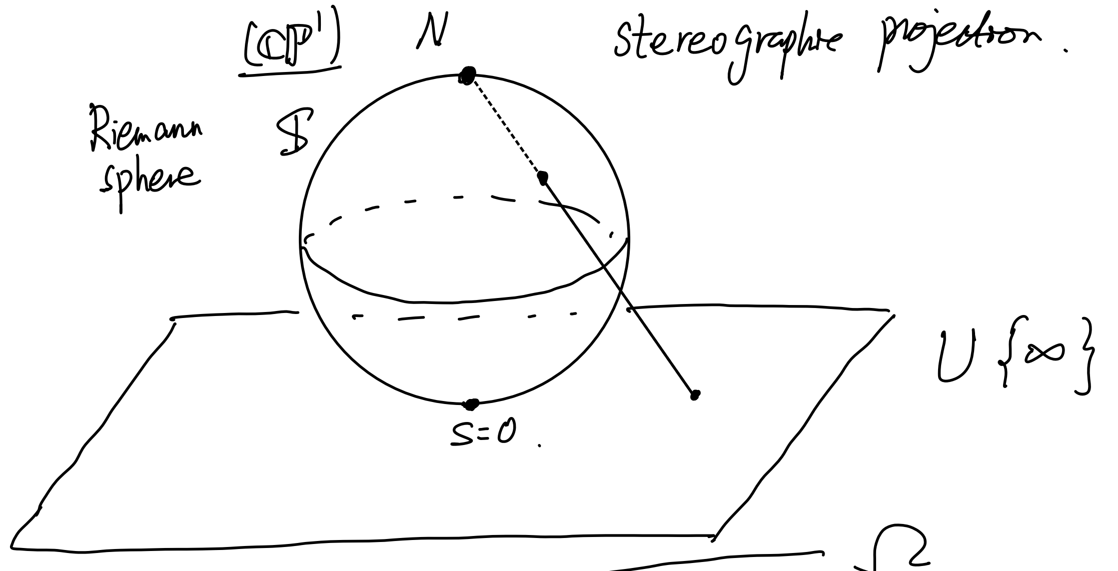
Ex: $f(z)=z^n+a_{n-1}z^{n-1}+\dots+a_1z+a_0$.

$$F(z)=f\left(\frac{1}{z}\right)=\frac{1}{z^n}+a_{n-1}\cdot\frac{1}{z^{n-1}}+\dots+\frac{a_1}{z}+a_0$$

$$=\frac{1}{z^n}\cdot\left(1+a_{n-1}z+\dots+a_1\cdot z^{n-1}+a_0\cdot z^n\right)$$

∞ is a pole of order n

Ex: $f(z)=e^z$. $F(z)=f\left(\frac{1}{z}\right)=e^{\frac{1}{z}}$ $\Rightarrow$ ∞ is an essential sing. for f.



$\mathbb{S} \longleftrightarrow \mathbb{C}\cup\{\infty\} \xleftarrow{f} \Omega$

- meromorphic functions (holomorphic functions with poles)

$\Updownarrow$

holomorphic map to the Riemann sphere $\mathbb{S}$.

- $\underline{\infty \text{ is not essential}}$ for holomorphic function $f: \mathbb{C}\setminus D_R(0) \to \mathbb{C}$

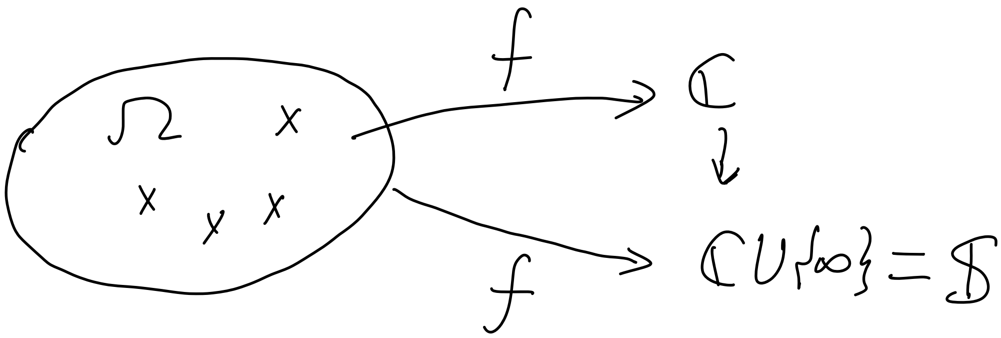
$\Updownarrow$

f extends to a holomorphic map $F: \mathbb{S}\setminus D_R(0) \to \mathbb{S}$.

$$\mathbb{S}\setminus D_R(0) = (\mathbb{C}\cup\{\infty\})\setminus D_R(0) \to \mathbb{C}\cup\{\infty\}$$

∞ is removable : $F(\infty) \neq \infty$

is pole : $F(\infty) = \infty$.



(at worst poles $\Longleftrightarrow$ f is either bounded or approaches ∞)

$\Updownarrow$

<u>Thm:</u> meromorphic functions on $\underline{\mathbb{C}\cup\{\infty\}}$

$\Updownarrow \quad \Uparrow$

rational functions.

$$\frac{P(z)}{Q(z)} = \frac{P(z)}{c_0(z-z_1)\cdots(z-z_m)}$$ has at worst poles at $z_1, \cdots, z_m$

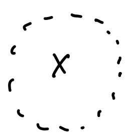
near ∞, $\dfrac{P(\frac{1}{z})}{Q(\frac{1}{z})} = \dfrac{\tilde{P}(z)}{\tilde{Q}(z)}$ has at worst pole at 0.

$\Rightarrow$ rational functions are meromorphic on $\mathbb{C}\cup\{\infty\}$

$$\left(\frac{a_n\left(\frac{1}{z}\right)^n + a_{n-1}\left(\frac{1}{z}\right)^{n-1} + \cdots + a_1\frac{1}{z} + a_0}{b_m\left(\frac{1}{z}\right)^m + b_{m-1}\left(\frac{1}{z}\right)^{m-1} + \cdots + b_1\frac{1}{z} + b_0} = \frac{\tilde{P}(z)}{\tilde{Q}(z)} \right)$$

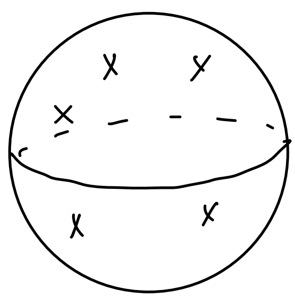
Ex: $\dfrac{e^{z}}{z}$ $\qquad \dfrac{e^{\frac{1}{z}}}{\frac{1}{z}} = z\cdot e^{\frac{1}{z}}$ has essential sing. at ∞

Pf: meromorphic on $\mathbb{C}\cup\{\infty\} \Rightarrow$ rational



$f(z)=(z-z_0)^k\cdot g(z) \qquad g(z_0)\neq 0$

zeros and poles are always isolated.



$= \mathbb{C}\cup\{\infty\} \Rightarrow$ finitely many poles.