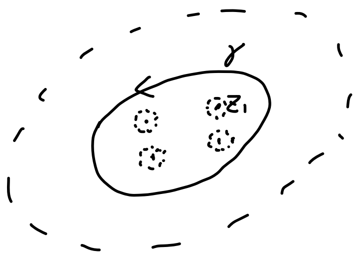
Residue Theorem:



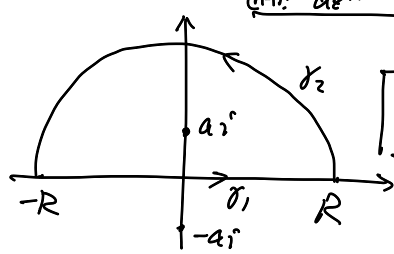
$$\underline{2\pi i \cdot \sum_i \int_{C_\varepsilon^i} f(z)\,dz}$$
$$\parallel$$
$$\int_\gamma f(z)\,dz = 2\pi i \cdot \sum_{i=1}^{N} \mathrm{res}_{z_i} f.$$

z_i: pole of f.
$$\underline{f(z) = \frac{a_{-n}}{(z-z_i)^n} + \cdots + \frac{a_{-2}}{(z-z_i)^2} + \frac{a_{-1}}{z-z_i} + G(z).}$$

$$a_{-1} = \mathrm{res}_{z_i} f$$
$$\parallel$$
$$\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_i)^n \cdot f\Big|_{z=z_i}$$
$$\parallel \; n=1$$
$$\boxed{\lim_{z\to z_i}\big((z-z_i)f(z)\big)}$$

Ex: $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+a^2}\,dx$. $a>0$

$e^{ix} = \cos x + i\sin x$



$$f(z) = \frac{e^{iz}}{z^2+a^2} = \frac{e^{iz}}{(z-ai)(z+ai)}.$$

$$\mathrm{res}_{ai} f = \frac{e^{i\cdot ia}}{ia+i\cdot a} = \frac{e^{-a}}{2ai}.$$

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{\overset{\cos x + i\cdot\underline{\sin x}}{\overset{\parallel}{e^{ix}}}}{x^2+a^2}\,dx = \int_{-R}^{R} \frac{\cos x}{x^2+a^2}\,dx$$

γ_2: $\theta \mapsto \underset{\underset{z(\theta)}{\parallel}}{R\cdot e^{i\theta}}$, $\underline{0 \le \theta \le \pi}$. $\quad dz = Re^{i\theta}\cdot i\,d\theta$

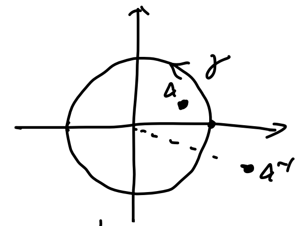
$$e^{iRe^{i\theta}} = e^{iR(\cos\theta + i\sin\theta)} = e^{-R\sin\theta}\,\underline{e^{iR\cos\theta}}$$

$$\int_{\gamma_2} f(z)\,dz = \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{\underset{\underset{\frac{1}{R}\cdot\text{const.}}{\wedge\!\wedge}}{R^2e^{2i\theta}+a^2}}\,\underset{\underset{dz}{\parallel}}{\underline{Re^{i\theta}\,i\,d\theta}} \xrightarrow{R\to+\infty} 0.$$

$$\lim_{R\to+\infty}\int_0^{\pi} \underline{e^{-R\sin\theta}}\,d\theta = 0$$

$$\Rightarrow \underset{\underset{2\pi i\cdot\frac{e^{-a}}{2ai}}{\parallel}}{\int_\gamma f(z)\,dz} \xrightarrow{R\to+\infty} \int_{-\infty}^{+\infty}\frac{\cos x}{x^2+a^2}\,dx \Rightarrow \int_{-\infty}^{+\infty}\frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}.$$

Ex: $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2a\cdot\cos\theta+a^2}$ $\quad |a|<1.$



$\gamma=\{|z|=1\}$: $\theta \mapsto e^{i\theta} = z(\theta) = \cos\theta + i\sin\theta$

$$dz = e^{i\theta}\cdot i\cdot d\theta = i\cdot z\cdot d\theta. \qquad \cos\theta = \frac{z+\bar z}{2} = \frac{z+z^{-1}}{2} \qquad |z|^2 = 1 = z\cdot\bar z$$

$$\sin\theta = \frac{z-z^{-1}}{2i}.$$

$$\int_\gamma f(z)dz = \int_\gamma \frac{\frac{dz}{iz}}{1-2a\cdot\frac{z+z^{-1}}{2}+a^2} = \frac{1}{i}\int_\gamma \frac{dz}{z-a(z^2+1)+a^2z} = i\int_\gamma \frac{dz}{az^2-(a^2+1)z+a}.$$

poles: $az^2-(a^2+1)z+a=0 \Rightarrow z = \dfrac{a^2+1\pm\sqrt{(a^2+1)^2-4a^2}}{2a} = \dfrac{(a^2+1)\pm\sqrt{(a^2-1)^2}}{2a}$

$$az^2-(a^2+1)z+a = (az-1)\cdot(z-a)$$

$$\begin{array}{cc} a & -1 \\ 1 & -a \end{array}$$

$$\frac{(a^2+1)\pm\sqrt{(a^2-1)^2}}{2a} = a,\ \frac{1}{a}. \qquad a = re^{i\theta},\quad \frac1a = r^{-1}e^{-i\theta}$$

$$f(z) = \frac{i}{az^2-(a^2+1)z+a} = \frac{i}{a(z-a)(z-\frac{1}{a})}. \qquad \mathrm{res}_a f = \frac{i}{a(a-\frac{1}{a})} = \frac{i}{a^2-1}$$

$$\int_\gamma f(z)dz = 2\pi i\cdot\frac{i}{a^2-1} = 2\pi\cdot\frac{1}{1-a}.$$

$$\frac{P(z)}{Q(z)} = \frac{A(z)}{(z-z_1)^{n_1}} + \frac{A_2(z)}{(z-z_2)^{n_2}} + \cdots$$

(Partial fraction)

Ex: Prove: $\cot(\pi z) = \frac{1}{\pi}\left[\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}\right], \quad z \in \mathbb{C}\setminus\mathbb{Z}.$

$$\cot(\pi z) = \boxed{\frac{\cos(\pi z)}{\sin(\pi z)}}$$

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\underline{\sin^2(z) + \cos^2(z) = 1.} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin(z) = 0 \Leftrightarrow e^{iz} = e^{-iz} = e^{2iz} = 1 \Rightarrow 2iz = i\cdot 2\pi\cdot n, \ n \in \mathbb{Z}. \Rightarrow z = \pi n$$

$$1 = e^{a+ib} = e^{a}(\cos b + i \sin b) \Leftrightarrow \begin{cases} e^{a}\cos b = 1 \Rightarrow b = 2\pi\cdot n,\ a = 0. \\ e^{a}\sin b = 0 \Rightarrow \sin b = 0,\ b = k\cdot\pi \end{cases}$$

$$a + ib = i\cdot 2\pi n$$

$z_0 \in \mathbb{C}\setminus\mathbb{Z}$, Consider $f(z) = \frac{\cot(\pi z)}{z - z_0}$ ← poles at $z = n$, $n \in \mathbb{Z}$

$z_0 \neq n$

$f(z)$ has poles at $\underline{z_0}$, $n \in \mathbb{Z}$.

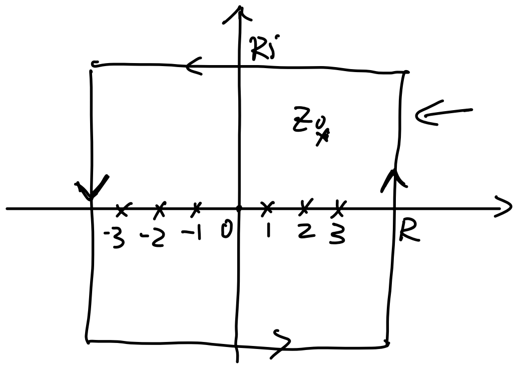
$$\operatorname{res}_{z_0} f = \cot(\pi z_0)$$

$$\underline{\operatorname{res}_n\left(\frac{\cot(\pi z)}{z - z_0}\right)} = \operatorname{res}_n\left(\frac{\cos \pi z}{z - z_0}\cdot\frac{1}{(-1)^n\cdot\pi(z-n)(1 - O(z-n))}\right) = \frac{\cos(\pi n)}{\pi(n - z_0)}\cdot(-1)^n = \frac{1}{\pi(n-z_0)}$$

$$\frac{1}{z - z_0}\cdot\frac{\cos(\pi z)}{\sin(\pi z)}$$

$$\sin(\pi z) = \sin(\pi z - \pi n)\cdot(-1)^n$$

$$\sin(\pi z - \pi n) = \pi\cdot(z-n) - \frac{\pi^3(z-n)^3}{3!} + \cdots = \pi(z-n)\cdot(1 - O(|z-n|))$$



$$z(t) = R + it, \quad -R \le t \le R$$

$$|f(z)| = \frac{|\cot(\pi z)|}{|z - z_0|} \sim R$$

$$R + it - z_0$$

$$\frac{-2z_0}{z_0^2 - n^2} = \frac{1}{-n - z_0} + \frac{1}{n - z_0}$$

$$\int_{\gamma_R} \underset{\dfrac{\cot(\pi z)}{z - z_0}}{\underset{\shortparallel}{f(z)}}\, dz = 2\pi i \cdot \sum_i \mathrm{res}_{z_i} f = 2\pi i \cdot \left[\overset{\mathrm{res}_{z_0} f}{\overset{\shortparallel}{\cot(\pi z_0)}} + \underbrace{\sum_{-R < n < R} \overset{\mathrm{res}_n f}{\overset{\shortparallel}{\frac{1}{\pi (n - z_0)}}}}_{-\frac{1}{\pi} \sum_{n=1}^{\lfloor R \rfloor} \frac{2z_0}{z_0^2 - n^2} - \frac{1}{\pi z_0}} \right]$$

$$\cot(\pi z_0) = \frac{1}{\pi} \cdot \left[\frac{1}{z_0} + \sum_{n=1}^{\infty} \frac{2z_0}{z_0^2 - n^2} \right]$$

$$|\cot(\pi(R + it))| = \frac{|\cos \pi(R + it)|}{|\sin \pi (R + it)|}$$

$$\underset{1 - \sin^2 a}{\underset{\shortparallel}{\cos^2 a}} \cdot ch^2(b) + \sin^2 a \cdot sh^2(b) = ch^2(b) - \sin^2 a \cdot \overset{1}{(ch^2 b - sh^2 b)}$$

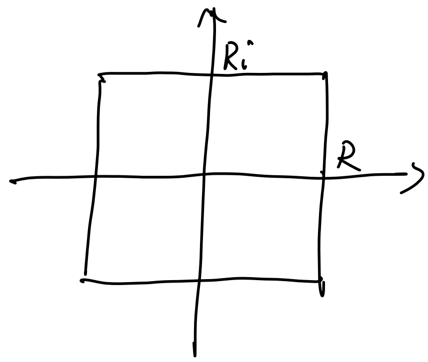
$$\cos(a + ib) = \cos a \cdot \cos(ib) - \sin(a) \cdot \sin(ib) = \cos a \cdot ch(b) - i \cdot \sin a \cdot sh(b).$$

$$\cos(ib) = \frac{e^{i \cdot ib} + e^{-i \cdot ib}}{2} = \frac{e^{-b} + e^{b}}{2} = ch(b)$$

$$\sin(ib) = \frac{e^{i \cdot ib} - e^{-i \cdot ib}}{2i} = \frac{e^{-b} - e^{+b}}{2i} = i \cdot \frac{e^{b} - e^{-b}}{2} = i \cdot sh(b)$$

$|\cot(\pi(R+it))| \le 2.$

on γ_R: $|\cot(\pi z)| \le 2 \; (R \gg 1)$



$$f(z) = \frac{\cot(\pi z)}{z - z_0}$$

$$\int_{\gamma_R} \frac{\cot(\pi z)}{z - z_0}\,dz = \underbrace{\int_{\gamma} \frac{\cot(\pi z)}{z}\,dz}_{= 2\pi i \cdot \sum_{\substack{k=-n \\ k \neq 0}}^{n} \frac{1}{\pi k} \; = 0} + \int_{\gamma_R} \frac{z_0 \overbrace{\cot(\pi z)}^{\le 2}}{z(z - z_0)}\,dz \;\xrightarrow{R \to +\infty} 0$$

$\le \frac{1}{R^2} \cdot 4R$

$\downarrow R \to +\infty$

0

$\frac{1}{\pi z^2} + G(z) \leftarrow$ hol.

$= \frac{1}{\pi z^2} \cdot \left(1 + z^2 \cdot \pi^2 \left(-\frac{1}{2} + \frac{1}{6}\right) + O(z^3)\right)$

$g = \frac{\cot(\pi z)}{z}$: pole at 0, $n \neq 0 \in \mathbb{Z}$

$\mathrm{res}_0\, g = 0$

$1 - \frac{1}{2}(\pi z)^2 + \frac{1}{4!}(\pi z)^4$

$$g(z) = \frac{\cot(\pi z)}{z} = \frac{\cos(\pi z)}{z \sin(\pi z)} = \frac{\cos(\pi z)}{z \cdot \pi z}\left(1 + \frac{(\pi z)^2}{3!} + O(z^3)\right)$$

$\sin(\pi z) = \pi z \left(1 - \frac{(\pi z)^2}{3!} + O(z^4)\right)$

$$\mathrm{res}_0\, g = \lim_{z \to 0} \left(\frac{d}{dz} z^2 g\right)$$

$$\frac{1}{1-\varepsilon} = 1 + \varepsilon + \varepsilon^2 + \cdots$$

$$\lim_{z \to 0} \frac{d}{dz}\left(z^2 \cdot \frac{\cot(\pi z)}{z} = \left(\frac{z \cdot \cos(\pi z)}{\sin(\pi z)}\right)\right) = 0.$$

$$\frac{\cancel{z} \cdot \left(1 - \frac{\pi^2 z^2}{2!} + O(z^3)\right)}{\pi \cancel{z} \left(1 - \frac{(\pi z)^2}{3!} + \cdots\right)} = \frac{1}{\pi} \cdot (1 + O(z^2))$$

$$\boxed{\frac{\pi^2}{(\sin \pi u)^2} = \sum_{n=-\infty}^{+\infty} \frac{1}{(u+n)^2} \qquad f(z) = \frac{\pi \cot(\pi z)}{(u+z)^2}}$$

Formula: $\cot(\pi z) = \frac{1}{\pi}\left[\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}\right] \quad z \in \mathbb{C}\setminus\mathbb{Z}.$

$$\Rightarrow \pi z \cdot \cot(\pi z) = 1 + \sum_{n=1}^{\infty} \frac{2z^2}{z^2-n^2}$$

$$= 1 - \sum_{n=1}^{\infty} \frac{2z^2}{n^2\left(1-\frac{z^2}{n^2}\right)}$$

$$= 1 - \sum_{n=1}^{\infty} \frac{2z^2}{n^2} \sum_{k=0}^{\infty} \left(\frac{z^2}{n^2}\right)^k \qquad z^{2+2k} = z^{2(k+1)}$$

$$1 - \sum_{k=1}^{\infty} \frac{2^{2k} B_k}{(2k)!} (\pi z)^{2k} \;\; \| \qquad = 1 - \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{2z^{2k}}{n^{2k}} = 1 - \sum_{k=1}^{\infty} \underbrace{\left(2\sum_{n=1}^{\infty} \frac{1}{n^{2k}}\right)}_{= 2\zeta(2k)} z^{2k}$$

$$\pi z \cdot \frac{\cos(\pi z)}{\sin(\pi z)} = \pi z \cdot \frac{1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \cdots}{\pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \cdots} = 1 - \frac{1}{3}(\pi z)^2 - \frac{1}{45}(\pi z)^4 - \frac{32}{3 \times 7!}(\pi z)^6 - \cdots$$

Bernouli numbers.

$$z \cdot \cot(z) = 1 - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} z^{2n}.$$

$$\Rightarrow \boxed{\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{2k}}}_{\zeta(2k)} = 2^{2k-1} \frac{\pi^{2k} B_k}{(2k)!}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{(Euler)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$