Morera's Thm: f continuous fct. in Ω (simply connected). If for any triangle $T \subset \Omega$, $\int_T f(z)\,dz = 0$. Then f is holomorphic.

Pf: Construct $F(z) = \int_{z_0}^{z} f(w)\,dw$. well-defined

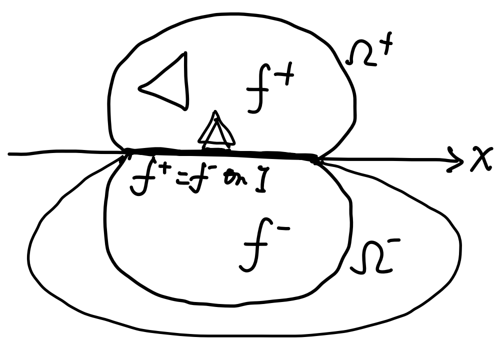
$F'(z) = f(z) \Rightarrow f(z)$ is holomorphic.

$\int_T f(z)dz = 0$

- $\{f_n(z)\}_{n=1}^{\infty}$ holomorphic $\to f(z)$ locally uniformly $\Rightarrow f(z)$ holomorphic

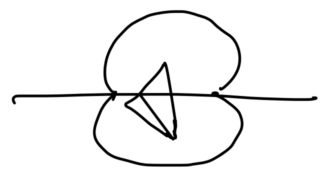
$f_n'(z) \to f'(z)$ locally uniformly.

- Thm (Symmetry principle)



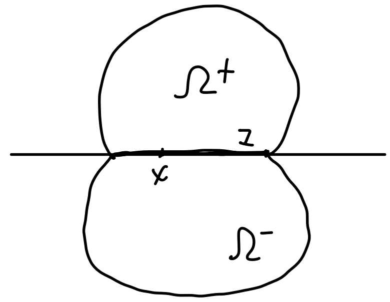
- $f^{\pm}$ holo. on $\Omega^{\pm}$
- $f^{\pm}$ extend continuously to I s.t. $f^+(x) = f^-(x)$ on I

$$f(z) = \begin{cases} f^+(z) & z \in \Omega^+ \\ f^+(z) = f^-(z) & z \in I \\ f^-(z) & z \in \Omega^- \end{cases}$$ is holomorphic



Pf: $\int_T f(z)dz = 0$ for any triangle. true $\Rightarrow f(z)$ holomorphic

Thm (Schwarz)

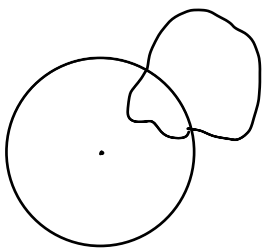


- f hol. on Ω^+
- f extends continuously to $I \subset \mathbb{R}$ s.t. $f(x) \in \mathbb{R}$

Then

$$F(z) = \begin{cases} f(z) & z \in \Omega^+ \\ f(z) = \overline{f(\bar{z})}, & z \in I \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}$$

is holomorphic on $\Omega = \Omega^+ \cup I \cup \Omega^-$

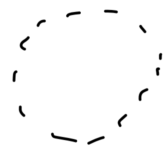


$$\lim_{h \to 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\bar{z})}}{h} = \lim_{h\to 0} \overline{\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}}$$

$$\downarrow$$

$$\overline{f'(\bar{z})}$$

$z \in \Omega^- \Rightarrow \bar{z} \in \Omega^+$

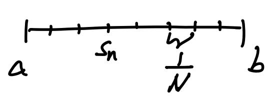


- $f(z) = \int_a^b F(z,s)\, ds$,
  - $F: \Omega \times [0,1] \to \mathbb{C}$ continuous
  - $\forall s \in [0,1]$, $F(z,s)$ is holomorphic w.r.t. z.

$\Rightarrow f(z)$ is holomorphic in Ω.

Pf:
- $\int_T f(z)dz = \int_T \int_a^b F(z,s)ds\, dz = \int_a^b \left(\int_T F(z,s)dz\right) ds = 0.$

- $$\underset{\underset{\displaystyle f(z)}{\parallel}}{\int_a^b F(z,s)ds} = \lim_{N\to\infty} \underset{\underset{\displaystyle f_N(z).}{\parallel}}{\left(\sum_{n=1}^{N} F(z,s_n)\cdot \frac{1}{N}\right)}$$



Show $f_N(z) \to f(z)$ locally uniformly $\Rightarrow f(z)$ holomorphic.

f holomorphic

- zero point $z_0 \in \Omega$, $f(z_0)=0$.

$$f(z) = 0 + a_1\cdot(z-z_0) + a_2\cdot\frac{(z-z_0)^2}{2!} + \cdots$$

$$= a_k\cdot(z-z_0)^k + a_{k+1}\cdot(z-z_0)^{k+1} + \cdots$$

$$= (z-z_0)^k\cdot\left(a_k + a_{k+1}(z-z_0) + a_{k+2}\cdot(z-z_0)^2 + \cdots\right) = (z-z_0)^k\cdot g(z)$$

$g(0) = a_k \neq 0$.

k: order of zero point z_0. (order of vanishing of z_0).

- z_0 is a pole of $f(z)$ if f is defined in punctured nbhd. U of z_0 and $\frac{1}{f(z)}$ defined to be 0 at z_0 is holomorphic in U

$$g(z) = \frac{1}{f(z)} = (z-z_0)^n\cdot h(z) \Rightarrow f(z) = \frac{1}{(z-z_0)^n h(z)} = \frac{1}{(z-z_0)^n}\left(\frac{1}{h(z)}\right) \quad n>0.$$

$(n>0)$, $h(z_0)\neq 0$

$n =$ order of pole z_0

$$= \frac{1}{(z-z_0)^n}\cdot\left(b_0 + b_1\cdot(z-z_0) + \cdots + b_n(z-z_0) + \cdots\right)$$

$$= \boxed{\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}} + \left(a_0 + a_1(z-z_0) + \cdots\right)$$

principal part of f at z_0

$$\lim_{z\to z_0} |f(z)| = \lim_{z\to z_0} \frac{1}{|z-z_0|^n}\left|\frac{1}{h(z)}\right| = +\infty.$$

$\frac{1}{|z-z_0|^n} \to +\infty$, $\left|\frac{1}{h(z)}\right| \neq 0$

$$\boxed{a_{-1} = \mathrm{Res}_{z_0} f(z)}$$

$a_{-n}\cdot(z-z_0)^{-n} + a_{-(n-1)}(z-z_0)^{-(n-1)} + \cdots + a_{-1}(z-z_0)^{-1}$
+
power series.

$$\boxed{\mathrm{Res}_{z_0} f(z) = \left.\left(\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n\cdot f(z)\right)\right|_{z=z_0} \Big/ (n-1)!}$$

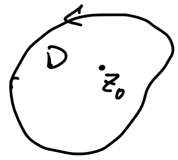
$(z-z_0)^n\cdot f(z) = a_{-n} + a_{-(n-1)}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} + \cdots$

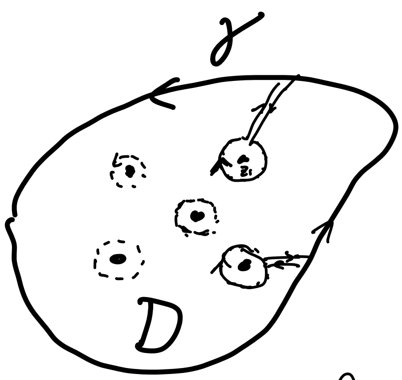
Residue Formula: f holomorphic in Ω except for poles at points $z_1, \cdots, z_N$

γ is a piecewise smooth closed curve that encloses a simply connected region D. No poles are contained in γ.

Then
$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{z_i \in D} \mathrm{res}_{z_i} f}$$

$$\Bigg(\underbrace{\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\,d\zeta}_{F(\zeta)} = 2\pi i \overset{\mathrm{res}_{z_0} F}{\overset{\shortparallel}{f(z_0)}} \qquad F(z) = \frac{f(z)}{z - z_0} \qquad \mathrm{Res}_{z_0} F(z) = f(z_0) \Bigg)$$





$$\int_\gamma f(z)\,dz = \sum_{z_i \in D} \int_{C_\varepsilon(z_i)} f(\zeta)\,d\zeta.$$

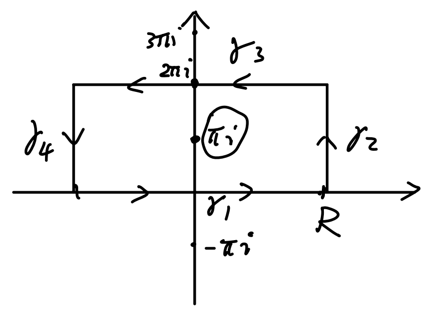
$$f(z) = \frac{a_{-n}}{(z - z_1)^n} + \frac{a_{-(n-1)}}{(z - z_1)^{n-1}} + \cdots + \boxed{\frac{a_{-1}}{z - z_1}} + \underset{\text{hol. near } z_1}{G_1(z)}$$

$$\int_{C_\varepsilon(z_1)} \frac{1}{(\zeta - z_1)^k}\,d\zeta = \begin{cases} C \cdot 1^{(k-1)} = 0, & k \neq 1,\ k > 1 \\ 2\pi i & k = 1. \\ 0 & k \leq 0 \end{cases}$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \qquad \int_{C_\varepsilon(z_1)} f(z)\,dz = 2\pi i \overset{\mathrm{res}_{z_1} f}{\overset{\shortparallel}{(a_{-1})}}$$

Def: f is called meromorphic if f is holomorphic except at poles. $\left(\overset{\Updownarrow}{\text{holomorphic map to } \mathbb{C}\cup\{\infty\}} \right)$

Ex: $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{x}}\,dx \quad (0<a<1)$



$$f(z)=\frac{e^{az}}{e^{z}+1} \qquad |e^{z}|=|e^{x+iy}|=e^{x}\neq 0.$$

$$e^{z}+1=0 \Leftrightarrow e^{z}=-1=e^{\pi i} \Rightarrow z=\pi i+(2\pi i\cdot k)$$

$$e^{z}+1=e^{z}-e^{\pi i}=\underbrace{\left(e^{(z-\pi i)}-1\right)}_{\substack{\| \\ \cancel{1}+(z-\pi i)+\frac{1}{2}(z-\pi i)^2 \cdots \cancel{-1}}}\cdot e^{\pi i}=\boxed{(z-\pi i)}\,e^{\pi i}\cdot\left(1+(z-\pi i)g(z)\right).$$

$$f(z)=\frac{e^{az}}{\boxed{(z-\pi i)}\,e^{\pi i}\cdot\left(1+(z-\pi i)g(z)\right)} \qquad \boxed{\mathrm{res}_{\pi i}f=\frac{e^{a\cdot\pi i}}{\underbrace{e^{\pi i}}_{=-1}}=-e^{a\pi i}}$$

- $\displaystyle\int_{\gamma_1} f(\zeta)\,d\zeta=\int_{-R}^{R} f(x)\,dx=\int_{-R}^{R}\frac{e^{ax}}{e^{x}+1}\,dx.$ $\qquad f(z)=\dfrac{e^{az}}{1+e^{z}}$

- $\gamma_2: t\mapsto \underbrace{R+it}_{z}, \quad 0\le t\le 2\pi$

$$\int_{\gamma_2} f(\zeta)\,d\zeta=\int_{0}^{2\pi}\frac{e^{a\cdot(R+it)}}{1+e^{R+it}}\,i\cdot dt \xrightarrow{R\to+\infty} 0.$$

$$\int_{\gamma_4} f(\zeta)\,d\zeta \xrightarrow{R\to+\infty} 0$$

$$\int_{\gamma_3} f(z)\,dz \qquad \gamma_3: t \mapsto t+2\pi i. \qquad t: R \to -R$$

$$\parallel$$

$$\int_R^{-R} \frac{e^{a(t+2\pi i)}}{1+e^{t+2\pi i}}\,dt = -\int_{-R}^{R} \frac{e^{at}\cdot e^{a\cdot 2\pi i}}{1+e^t}\,dt$$

$$\int_{\partial D} f(z)\,dz = \underbrace{(1-e^{a\cdot 2\pi i})\cdot \int_{-R}^{R} \frac{e^{at}}{1+e^t}\,dt}_{\downarrow \; I} + \underbrace{\int_{\gamma_2} + \int_{\gamma_4}}_{\downarrow R\to+\infty \; 0}$$

$$\parallel$$

$$2\pi i \cdot \mathrm{res}_{\pi i} f$$

$$\parallel$$

$$2\pi i \cdot (-e^{a\pi i})$$

$R \to +\infty$: $\quad 2\pi i\cdot(-e^{a\pi i}) = (1-e^{a\cdot 2\pi i})\cdot I$

$$\Rightarrow \quad I = \frac{2\pi i\, e^{a\pi i}}{-1+e^{a\cdot 2\pi i}} = \frac{\pi}{\sin \pi a}.$$

$$\parallel \qquad /\!/$$

$$\frac{\pi}{\frac{e^{a\pi i} - e^{-a\pi i}}{2i}}$$