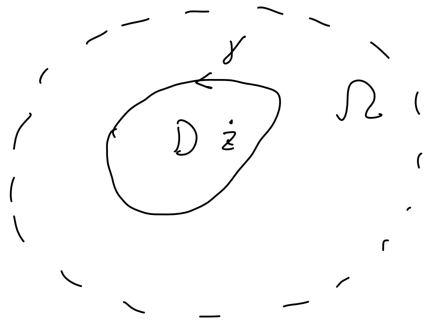
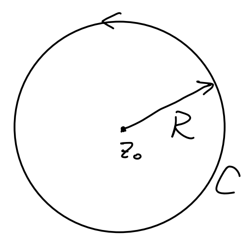
Cauchy integral formula:



$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.}$$



$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\cdot\int_{|\zeta - z_0| = R} \frac{|f(\zeta)|}{R^{n+1}}|d\zeta|$$

$$\leq \left(\max_{|\zeta - z_0| = R}|f(\zeta)|\right)\cdot\frac{n!}{2\pi}\cdot\frac{2\pi R}{R^{n+1}} = \underline{\frac{n!\,\max_C|f(\zeta)|}{R^n}}.$$

Thm (Liouville) f holomorphic on $\mathbb{C}$. if f is bounded, then $f \equiv$ constant.

Thm (fundamental Thm of algebra) $P(z)$ non-constant polynomial with $\mathbb{C}$-coefficients always has a root $z_1 \in \mathbb{C}$. As a consequence, $P(z) = a_0\cdot(z-z_1)\cdot(z-z_2)\cdots(z-z_n)$ where $n = \deg P(z)$.

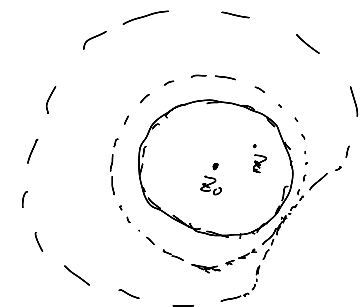
Pf: Consider $f(z) = \frac{1}{P(z)}$ and apply Liouville.

Thm: f holomorphic in Ω. $\overline{D} = \overline{D}(z_0, R)$ is a closed disk $\subset \Omega$.

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n\cdot(z - z_0)^n}$$ for all $z \in D$ and the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{for all } n \geq 0.$$

Pf: $f(z) = \frac{1}{2\pi i} \int_{\substack{C \\ \| \\ \partial D}} \frac{f(\zeta)}{(\zeta - z)} d\zeta$



$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}$$

$$\boxed{\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{R} < 1. \quad \zeta \in C}$$

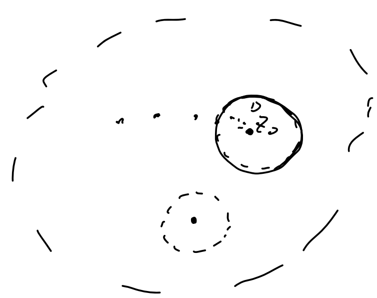
$$= \frac{1}{\zeta - z_0} \cdot \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^n}$$ converges uniformly for $\zeta \in C$

$$f(z) = \frac{1}{2\pi i} \int_C f(\zeta) \cdot \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta = \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i} \left(\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right)}_{a_n} \cdot (z - z_0)^n.$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \underset{\uparrow}{=} \frac{f^{(n)}(z_0)}{n!}$$

Cauchy integral formula for $f^{(n)}(z_0)$.

Thm: f holomorphic in Ω (connected region). f vanishes on a sequence of distinct pts with a limit point in Ω. Then f is identically 0. [In other words, the zeros of a non-zero hol. fct. are isolated].



Pf: $\boxed{\{w_k\}_{k=1}^{\infty} \text{ has } z_0 \text{ as limit point.}}$

$f(w_k) = 0, \ \forall k \Rightarrow \underset{\substack{\| \\ \lim_{z \to z_0} f(z)}}{f(z_0)} = 0$

$\boxed{\text{Suppose } f \not\equiv 0 \text{ inside } D}$ $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 \cdot (z - z_0) + a_2 \cdot (z - z_0)^2 + \cdots = a_k (z - z_0)^k (1 + g(z - z_0)).$

$\lim_{z \to z_0} g(z - z_0) = 0$

for $|z - z_0| < \varepsilon \ll 1$, $|1 + g(z - z_0)| \geq \delta > 0$

$\Rightarrow f(z)$ has no other zeros for $|z - z_0| < \varepsilon$. contradiction

$\Rightarrow$ $\underline{f \equiv 0 \text{ inside } D}$.

$U = \{z \in \Omega : \underline{f(z) = 0}\}^{\circ}$.

$\boxed{\begin{array}{r} \Omega \text{ connected.} \\ U \neq \phi. \quad U \text{ open. } \text{closed} \Rightarrow U = \Omega. \end{array}}$

$\Downarrow$

$f \equiv 0$ on Ω.

U

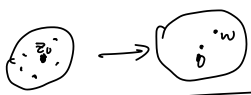
$z_n \to z_\infty$

$f(z_n) = 0 \Rightarrow f(z_\infty) = 0$. $\exists\ D_\varepsilon \ni z_\infty$ s.t. $f|_{D_\varepsilon} \equiv 0 \Rightarrow z_\infty \in U$

$f : \Omega \to \mathbb{C}$

$z_0 \longrightarrow 0$

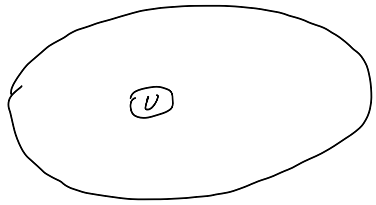
$(z - z_0)^{k}(1 + g)$ 2

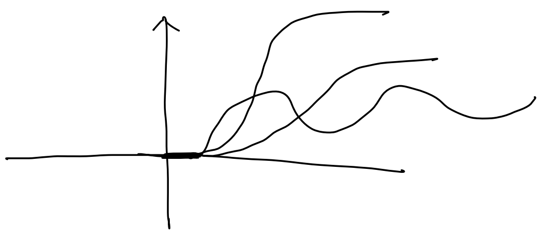


Cor: f, g hol. in Ω connected region.

$f(z) = g(z)$ for z in a non-empty open subset of Ω

$\Longrightarrow f \equiv g$ for all $z \in \Omega$.

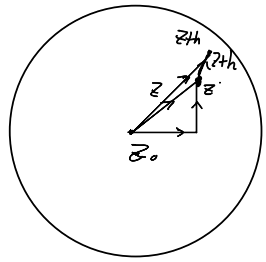




Morera's Thm (Converse to Cauchy's Thm).

Thm: Suppose f is continuous fct. in open disc D s.t. for any triangle T contained in D. $\int_T f(z)\,dz = 0$. Then f is holomorphic.

Pf: $\int_T f(z)\,dz = 0 \ \forall$ triangle T

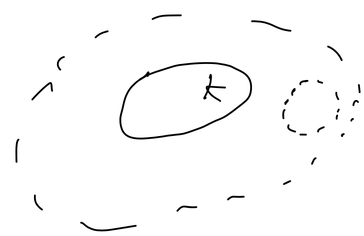


$\Rightarrow$ $f = F'$. $F(z) = \int_{z_0}^{z} f(w)\,dw$ holomorphic

$\Rightarrow$ $f = F'$ holomorphic.

$$\left(\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(w)\,dw \xrightarrow{h\to 0} f(z). \right).$$

Thm: $\{f_n\}_{n=1}^{\infty}$ seq. of hol. fct. that converges uniformly to a fct. f in every compact subset of Ω, then f is holomorphic.

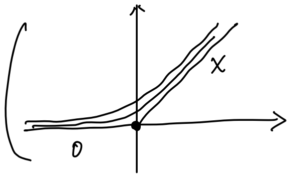


Pf: $\int_T f(z)\,dz = 0 \Rightarrow f$ is holomorphic.

$\uparrow$

for any triangle contained in $D \subset \Omega$. $\int_T f_n(z)\,dz = 0$

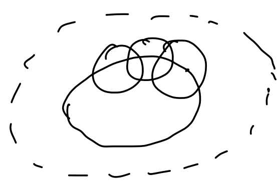
disk

(


not true for real case)

[locally uniformly]

Thm: Assumption as above thm, $\{f_n'\}$ converges to f' uniformly on every cpt. subset of Ω.



Pf: $\forall \overset{\text{Fix}}{\delta > 0}$, $\underline{D_\delta \subset \Omega}$. $\boxed{|f_n'(z) - f'(z)| < \varepsilon \text{ for } n \geq N.}$

$$\| \quad (f_n(z) - f(z))' = F'(z). \quad F(z) = f_n(z) - f(z)$$

$$|F'(z)| = \left| \frac{1}{2\pi i} \int_{\underset{C}{C(z,\delta)}} \frac{F(\zeta)}{(\zeta - z)^2} d\zeta \right|$$

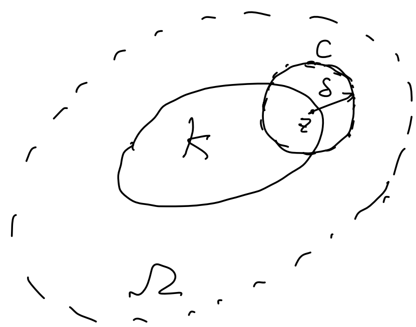
$$\wedge\|$$

$$\frac{1}{2\pi} \int_C \frac{|F(\zeta)|}{|\zeta - z|^2} |d\zeta|$$

$$\wedge\|$$

$$\max_C |F(\zeta)| \cdot \frac{1}{\delta^2} \frac{1}{2\pi} \cdot 2\pi\delta = \frac{1}{\delta} \quad < \varepsilon.$$

$$\max_C |f_n(z) - f(z)| < \varepsilon\delta \text{ for } n \geq N(\varepsilon).$$



So. $f_n^{(k)}(z) \longrightarrow f^{(k)}(z)$ Locally uniformly

(uniformly over any compact subset).

- $$\sum_{n=0}^{\infty} f_n(z) = F(z) = \lim_{N \to \infty} \underbrace{\sum_{n=0}^{N} f_n(z)}_{F_N(z)}.$$

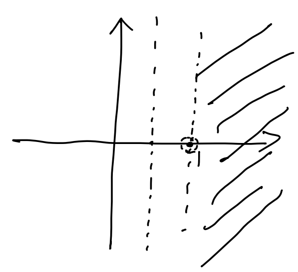
$$\zeta(z) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{n^z}}_{n^{-z}} =: f_n(z) \qquad n^z = n^{x+iy} = e^{z \cdot \ln n}$$

$$|n^{-z}| = |n^{-x} \cdot \underbrace{n^{-iy}}_{e^{-iy \cdot \log n}}| = n^{-x}.$$

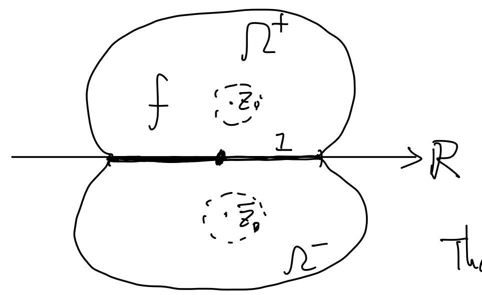
$|\zeta(z)| \leq \sum_{n=0}^{\infty} \frac{1}{n^x}$ ← convergent when $x = \text{Re } z > 1$.

$F_N(z) \longrightarrow \zeta(z)$ $\overset{\text{Locally}}{\text{uniformly}}$ on $\Omega = \{\text{Re } z > 1\}$.

$$\| \quad F_N(z) = \sum_{n=0}^{N} \frac{1}{n^z}$$



- Schwarz reflection principle.



- f is holomorphic in Ω^+
- f extends continuously to I s.t. f is real valued on I

Then there exists a function F holomorphic in $\Omega = \Omega^+ \cup I \cup \Omega^-$ s.t. $F|_{\Omega^+} = f$.

$$F(z) = \begin{cases} f(z) & z \in \Omega^+ \cup I \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}$$

$f(z) = \sum_n a_n (z - z_0)^n$

$\overline{f(\bar{z})} = \sum_n \bar{a}_n (z - \bar{z}_0)^n$

use Cauchy-Riemann to prove $F(z)$ is holomorphic in Ω^-

Morera