Cauchy's Thm: f holomorphic in Ω, $\gamma \subset \Omega$ that encloses a simply connected domain. Then $\int_\gamma f(z)\,dz = 0$.

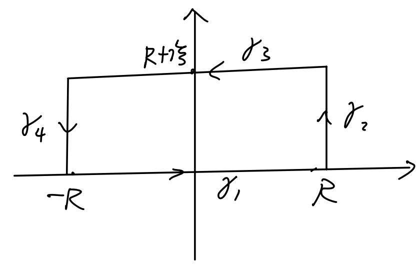
(no holes interior)

Example: $\xi \in \mathbb{R}$

$$\boxed{\int_{-\infty}^{+\infty} e^{-\pi x^2} \cdot e^{-2\pi i x \cdot \xi}\, dx}$$

$$\| $$

$$\boxed{\begin{array}{c} \int_{-\infty}^{+\infty} e^{-\pi x^2} \cdot \cos(2\pi x \xi)\, dx \\ + \\ -i \int_{-\infty}^{+\infty} e^{-\pi x^2} \sin(2\pi x \xi)\, dx \end{array}}$$



$$f(z) = e^{-\pi z^2} \quad \cdot \int_{\gamma_1} f(z)\,dz = \int_{-R}^{+R} e^{-\pi x^2} dx \xrightarrow{R\to+\infty} \int_{-\infty}^{+\infty} e^{-\pi x^2} dx = 1$$

$$\int_{-\infty}^{+\infty} e^{-\pi x^2} dx \cdot \int_{-\infty}^{+\infty} e^{-\pi y^2} dy = \iint_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\, dx dy$$

$$= \int_0^\infty dr \int_0^{2\pi} e^{-\pi r^2}\, r\, dr\, d\theta = \Big(\underbrace{\int_0^\infty e^{-\pi r^2} r\, dr}_{\|}\Big) \cdot \underbrace{\int_0^{2\pi} d\theta}_{\| \atop 2\pi}$$

$$\int_0^\infty e^{-t} dt \cdot 1 = -e^{-t}\Big|_0^\infty = 1. \qquad \int_0^\infty e^{-\pi r^2} \frac{d(\pi r^2)}{2} \cdot \frac{1}{\pi} \cdot 2\pi$$

$\cdot\ \gamma_2$: $t \mapsto \underbrace{R + i \cdot t}_{\| \atop z(t)}$. $\underline{0 \le t \le \xi}$. $dz = i\,dt$

$$\int_{\gamma_2} e^{-\pi z^2} dz = \int_0^{\xi} e^{-\pi \underbrace{(R+it)^2}_{\| \atop (R^2-t^2)+2iRt}} \cdot i\, dt = \int_0^{\xi} \Big(e^{-\pi((R^2-t^2)+2iRt)}\Big) i\, dt$$

$$\| $$

$$e^{-\pi(R^2-t^2)}\ e^{-2\pi i R t}$$

$$\downarrow R\to+\infty$$

$$0$$

$$\cdot \left|\int_{\gamma_4} e^{-\pi z^2} dz\right| \le \int_{\gamma_3} |e^{-\pi z^2}|\,|dz|$$

$$\wedge|$$

$$\max_{\gamma_3} |e^{-\pi z^2}| \cdot |\xi| \xrightarrow{R\to+\infty} 0$$

- γ_3: $t \mapsto z(t) = t + i\cdot\xi$, $\quad t: R \to -R$.

$$\int_{\gamma_3} f(z)\,dz = \int_R^{-R} e^{-\pi\cdot\underset{\substack{\parallel \\ (t^2-\xi^2)+2it\xi}}{(t+i\xi)^2}} dt = -\int_{-R}^{R} e^{-\pi\cdot\left((t^2-\xi^2)+2it\xi\right)} dt$$

$$= \underline{-\int_{-R}^{R} e^{-\pi t^2 - 2\pi i t\xi}\, dt} \cdot e^{\pi\xi^2}.$$

$$0 = \int_\gamma f(z)\,dz = \underset{\substack{\downarrow R\to+\infty \\ 1}}{\int_{\gamma_1}} + \underset{\substack{\downarrow R\to+\infty \\ 0}}{\int_{\gamma_2}} + \int_{\gamma_3} + \underset{\substack{\downarrow R\to+\infty \\ 0}}{\int_{\gamma_4}} \xrightarrow{R\to+\infty} 1 - \int_{-\infty}^{+\infty} e^{-\pi x^2 - 2\pi i x\xi} e^{\pi\xi^2}.$$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} e^{-\pi x^2 - 2\pi i x \xi}\, dx = e^{-\pi\xi^2}}$$

(Fourier transform of $\underline{e^{-\pi x^2}}$ is (itself) $\underline{e^{-\pi\xi^2}}$.)

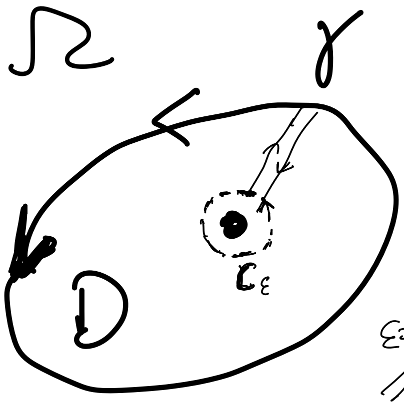
$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\cdot e^{-2\pi i\cdot x\cdot\xi}\, dx \quad \text{Fourier transform.}$$

- Cauchy integral formula

f holomorphic in Ω. $\gamma \subset \Omega$ closed curve that encloses a simply connected domain $D \subset \Omega$.

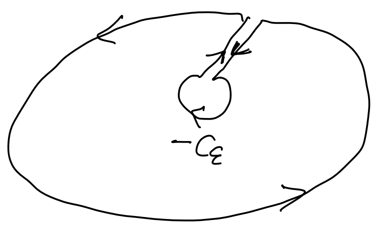
$$\boxed{f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta}$$

has a pole at $\zeta = z$.



$$\int_\gamma \frac{f(\zeta)}{\zeta-z}\,dz = \int_{C_\varepsilon}\frac{f(\zeta)}{\zeta-z}\,d\zeta$$

$\varepsilon \to 0$



$$C_\varepsilon:\ \theta \longrightarrow \overset{\zeta(\theta)}{\overset{\shortparallel}{z+\varepsilon e^{i\theta}}} \qquad 0\le\theta\le 2\pi$$

$$\int_{C_\varepsilon}\frac{f(\zeta)}{\zeta-z}\,d\zeta = \int_0^{2\pi}\frac{f(z+\varepsilon e^{i\theta})}{\cancel{\varepsilon e^{i\theta}}}\,\cancel{\varepsilon e^{i\theta}}\cdot i\,d\theta$$

$$= i\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta$$

$$\Big\downarrow\ \varepsilon\to 0$$

$$i\int_0^{2\pi} f(z)\,d\theta = i\cdot f(z)\cdot 2\pi.$$

$$\left(\left|\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta-\int_0^{2\pi} f(z)\,d\theta\right| \le \int_0^{2\pi}\Big|\underbrace{f(z+\varepsilon e^{i\theta})-f(z)}_{\overset{\shortparallel}{f'(z)\cdot\varepsilon e^{i\theta}+o(\varepsilon)}}\Big|\,d\theta \xrightarrow{\varepsilon\to 0} 0\right)$$

$$\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta = 2\pi i\cdot f(z) \iff \boxed{f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta}.$$

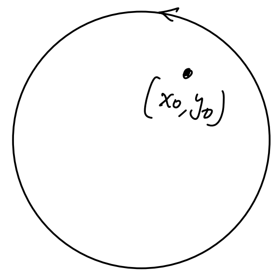
$$\overset{\shortparallel}{\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{\zeta-z}\right) f(\zeta)\,d\zeta}$$

$$\boxed{\frac{\partial}{\partial\bar\zeta}\left(\frac{1}{2\pi i}\cdot\frac{1}{\zeta-z}\right)=\delta_z}$$

$u(x,y)$ is harmonic if $\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0$.

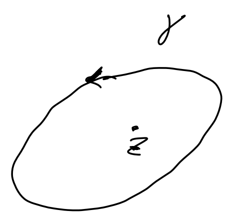
$$u(x_0,y_0) = \int_{S_R} P(x_0,y_0;\, x,y) \cdot u(x,y)\, ds$$

$\uparrow$ Poisson kernel



$$\boxed{\Delta_{x,y} P(x_0,y_0;x,y) = \delta_{x_0,y_0}.}$$

Cor: $$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$



$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_\gamma \underline{\left(\frac{f(\zeta)}{\zeta-(z+h)} - \frac{f(\zeta)}{\zeta - z}\right)\frac{1}{h}}\, d\zeta.$$

$$\Big\downarrow h\to 0 \qquad\qquad \Vert$$

$$f'(z) \qquad\qquad f(\zeta)\cdot\boxed{\left(\frac{1}{\zeta-(z+h)} - \frac{1}{\zeta-z}\right)\frac{1}{h}}\, d\zeta$$

$$\Big\downarrow h\to 0$$

$$\frac{1}{(\zeta-z)^2} = \frac{\partial}{\partial z}\left(\frac{1}{\zeta-z}\right).$$

$$\longrightarrow \underline{\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}\, d\zeta} = \underline{\frac{\partial}{\partial z}\left(\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta\right)}$$

$$\left| \int_\gamma \underline{f(\zeta)} \left(\left(\frac{1}{\zeta-(z+h)} - \frac{1}{\zeta-z} \right) \frac{1}{h} - \frac{1}{(\zeta-z)^2} \right) d\zeta \right| \xrightarrow{h\to 0} 0.$$

$$\wedge\!|$$

$$\underbrace{\max_\gamma |f(\zeta)|}_{\substack{\wedge\!| \\ M}} \cdot \underbrace{\max_\gamma \left| \left(\frac{1}{\zeta-(z+h)} - \frac{1}{\zeta-z} \right) \frac{1}{h} - \frac{1}{(\zeta-z)^2} \right|}_{\substack{\downarrow h\to 0 \\ 0}} \cdot L(\gamma)$$

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta-z)^2} d\zeta$$

$$\Rightarrow f''(z) = \frac{1}{2\pi i} \int_\gamma \frac{2}{(\zeta-z)^3} f(\zeta)\, d\zeta$$

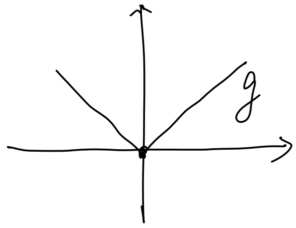
$$\Rightarrow f^{(n)}(z) = \frac{1}{2\pi i} \int_\gamma \frac{n!}{(\zeta-z)^{n+1}} f(\zeta) d\zeta = \underline{\frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}} d\zeta}.$$

$$f^{(n+1)}(z) = \lim_{h\to 0} \frac{f^{(n)}(z+h) - f^{(n)}(z)}{h} = \underbrace{\frac{n!}{(2\pi i)} \int_\gamma f(\zeta) \left[\frac{1}{(\zeta-(z+h))^{n+1}} - \frac{1}{(\zeta-z)^{n}} \right] \frac{1}{h} d\zeta}_{\downarrow h\to 0}$$

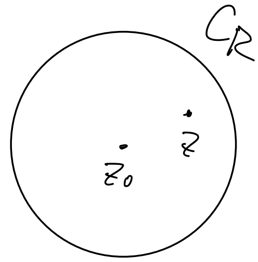
$$\frac{(n+1)!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}} d\zeta = \boxed{\frac{n!}{2\pi i} \int_\gamma f(\zeta)\, \underline{\frac{n+1}{(\zeta-z)^{n+1}}} d\zeta.}$$

$f^{(n)}(z)$ exists for all $n \in \mathbb{Z}_{\geq 0}$.

So. $\boxed{f \text{ is holomorphic} \Rightarrow f \text{ is infinitely differentiable!}}$



$f' = g$.



· Cor:

$$\left|f^{(n)}(z)\right| = \left|\frac{n!}{2\pi i}\int_{C_R} \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta\right|$$

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\int_{C_R}\left|\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\right||d\zeta| \qquad \zeta = z_0 + R\cdot e^{i\theta}$$

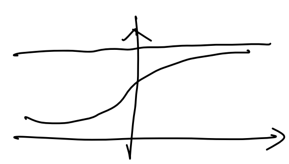
$$\le \frac{n!}{\cancel{2\pi}}\cdot \max_{C_R}|f(\zeta)|\cdot \underbrace{\int_0^{2\pi}\frac{R\,d\theta}{R^{n+1}}}_{=\frac{1}{R^n}\cdot\cancel{2\pi}} = \frac{n!}{R^n}\underbrace{\max_{C_R}|f(\zeta)|}_{=\|f\|_{C_R}}$$

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{R^n}\max_{C_R}|f(\zeta)| \qquad \text{(Cauchy inequality).}$$

Application (Liouville Thm): If f is $\underset{\uparrow}{\underline{\text{entire}}}$ and $\underset{\uparrow}{\underline{\text{bounded}}}$, then f is constant.

f is holomorphic on $\mathbb{C}$

$\max_{z\in\mathbb{C}}|f| \le M < +\infty$.



$$|f'(z_0)| \le \frac{1}{R} \underbrace{\max_{C_R} |f(\zeta)|}_{\le M < +\infty} \xrightarrow{R \to +\infty} 0 \quad \text{for any } z_0 \in \mathbb{C}$$

$$\Rightarrow f'(z) \equiv 0. \Rightarrow f(z) \text{ is constant} = f(0).$$

$$f(z) = \underline{\int_0^z f'(\zeta)\, d\zeta + f(0)}$$

Cor: <u>Any non-constant polynomial has a root over $\mathbb{C}$.</u>

<u>Any polynomial splits into product of linear factors.</u>

$$\left(P(z) = (z - z_1) \cdot P_1(z) = (z - z_1) \cdot (z - z_2) \cdot P_2(z) = \cdots = (z - z_1) \cdots (z - z_n) \cdot C \right)$$

Pf: $P(z)$ polynomial. <u>Suppose $P(z)$ does not a root over $\mathbb{C}$.</u>

$\boxed{f(z) = \frac{1}{P(z)} \text{ is entire}}$. Claim: $\left| \frac{1}{P(z)} \right| \le M < +\infty$ for any $z \in \mathbb{C}$.

$$P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n.$$

$$|P(z)| = |z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n|$$
$$\ge |z|^n - a_1 |z|^{n-1} - a_2 |z|^{n-2} - \cdots - |a_n|$$

$$\left|\frac{1}{p(z)}\right| = \left|\frac{1}{z^n\left(1+\frac{a_1}{z}+\cdots+\frac{a_n}{z^n}\right)}\right| \leq \frac{1}{|z^n|}\cdot\frac{1}{\left(1-\frac{|a_1|}{|z|}-\cdots-\frac{|a_n|}{|z|^n}\right)} \leq M < +\infty$$

when $|z|$ sufficiently large. $|z| \geq R >> 1$.

$$\left|1+\frac{a_1}{z}+\cdots+\frac{a_n}{z^n}\right| \geq 1-\left|\frac{a_1}{z}+\cdots+\frac{a_n}{z^n}\right| \geq 1-\frac{|a_1|}{|z|}-\cdots-\frac{|a_n|}{|z^n|}.$$

$$\left(|z_1+z_2| \geq |z_1|-|z_2|.\right)$$

over $\underset{\underset{D_R}{\shortparallel}}{\{|z|\leq R\}}$, $\max\limits_{D_R}\left|\frac{1}{p(z)}\right| < +\infty$.

$$\Rightarrow \sup_{z\in\mathbb{C}}\left|\frac{1}{p(z)}\right| < +\infty \xRightarrow{\text{Liouville}} \frac{1}{p(z)} = \text{constant}$$

$$\Rightarrow p(z) = \text{constant} \quad \text{contradiction.}$$