$\gamma: \quad z = z(t) \qquad a \le t \le b$

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

Thm (Cauchy Thm) If f is holomorphic. if γ encloses a simply connected domain D.

Then $\int_\gamma f(z)\,dz = 0$.

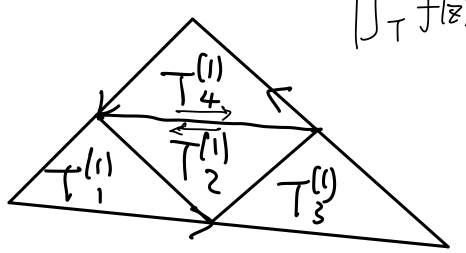
Pf (assuming $z \mapsto f'(z)$ is continuous) Stoke's Thm

$$\int_\gamma f(z)dz = \iint_D \underbrace{d(f(z)dz)}_{= \left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\right)\wedge dz} = \iint_D \frac{\partial f}{\partial \bar z}\, d\bar z \wedge dz \overset{f \text{ holomorphic}}{=\!=} 0.$$

Thm (Goursat's Thm) Ω open in $\mathbb{C}$. $T \subset \Omega$ triangle, interior $\subset \Omega$.

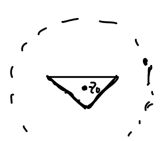
Then $\int_T f(z)dz = 0$.

$$\left|\int_T f(z)dz\right| = \left|\sum_{i=1}^4 \int_{T_i^{(1)}} f(z)\,dz\right| \le 4 \left|\int_{T_i^{(1)}} f(z)\,dz\right| \text{ for some } i$$



$$\rightsquigarrow \left|\int_T f(z)dz\right| \le 4^n \left|\int_{T^{(n)}} f(z)dz\right| \xrightarrow{n\to\infty} 0.$$

$T^{(n)} \sim 2^{-n}\cdot d$



$$f(z) = f(z_0) + f'(z_0)\cdot(z-z_0) + \underbrace{o(|z-z_0|)}_{=(z-z_0)\cdot\psi(z)} \qquad \psi(z) \xrightarrow{z\to z_0} 0$$

$$4^n\left|\int_{T^{(n)}} f(z)dz\right| = 4^n\left|\int_{T^{(n)}} (z-z_0)\cdot\psi(z)\,dz\right| \le 4^n \int_{T^{(n)}} |z-z_0|\,|\psi(z)|\,|dz|.$$

$$4^n\cdot 2^{-n}\cdot 2^{-n}\cdot 1\cdot \max_{T^{(n)}}|\psi(z)| \to 0$$

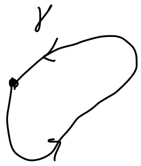
Fact: • If $\underline{f(z) = F'(z)}$ (i.e. f has a primitive), then for curve γ

$z = z(t),\ a \le t \le b$

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^b F'(z(t))\,z'(t)\,dt$$

γ



$$\int_a^b \underline{\frac{d}{dt} F(z(t))}\,dt = F(z(t))\Big|_{t=a}^{t=b} \underset{z(a)=z(b)}{\overset{\text{if } \gamma \text{ is closed}}{=\!=}} 0.$$

$1 = z'$ $\quad z^n = \left(\frac{1}{n+1} z^{n+1}\right)'$

• $\underline{\left|\int_\gamma f(z)\,dz\right|} = \left|\int_a^b f(z(t))\,z'(t)\,dt\right| \le \int_a^b |f(z(t))|\,|z'(t)|\,dt$

$$\le \max_\gamma |f| \cdot \underline{\int_a^b |z'(t)|\,dt} = \underline{\max_\gamma |f| \cdot L(\gamma)}$$

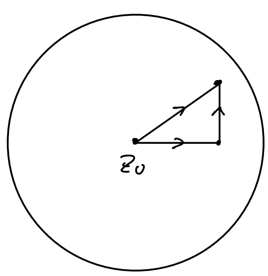
$$\int_a^b |z'(t)|\,dt = \underline{\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt} = \int_a^b ds$$

Cor: $\int_{\text{polygon}} f(z)\,dz = 0$.

Thm: A holomorphic fct. f in an open disc $\underline{\text{has a primitive}}$, i.e. $\underline{f = F'}$.

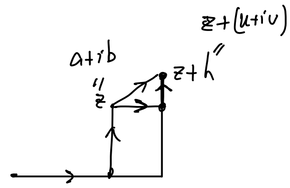
$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Pf:



$$\int_{z_0}^{z} f(z)\,dz = F(z).$$

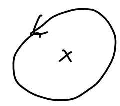
$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}$$



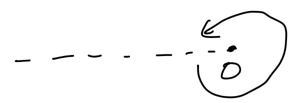
$$F(z+h)-F(z)=\int_z^{z+h} f(w)\,dw. \qquad f(z)=\frac{1}{h}\int_z^{z+h} f(z)\,dw.$$

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|=\left|\frac{1}{h}\int_z^{z+h}\big(f(w)-f(z)\big)\,dw\right|\leq\left(\frac{1}{|h|}\cdot|h|\right)\left(\max_w|f(w)-f(z)|\right)\to 0.$$

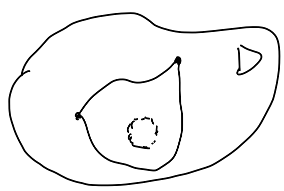
$$\Rightarrow\ F'(z)=f(z).$$



- $f(z)=\frac{1}{z}$. holomorphic on $\mathbb{C}\setminus\{0\}$. no primitive on $\mathbb{C}\setminus\{0\}$.

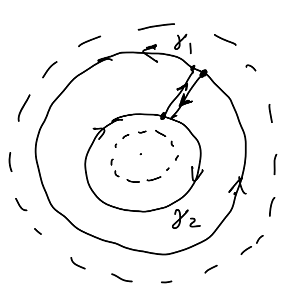


$$F(z)=\log z=\log(r\cdot e^{i\theta})=\log r+i\theta.$$

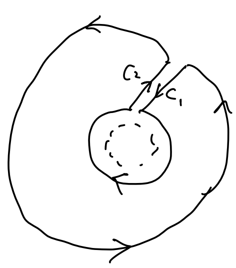


<u>Thm</u>: f is holomorphic in a disk D, γ is closed curve $\subset D$

Then $\int_\gamma f(z)\,dz=0$.



$$\boxed{\int_{\gamma_1} f(z)\,dz=\int_{\gamma_2} f(z)\,dz}$$

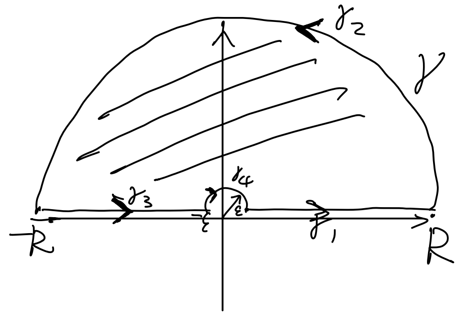


$$\int_{\gamma'} f(z)\,dz=0=\int_{\gamma_1} f(z)\,dz-\int_{\gamma_2} f(z)\,dz$$

$$\|$$

$$\underline{\int_{\gamma_1'}}+\underline{\int_{c_1}}+\int_{-\gamma_2'}+\underline{\int_{c_2}}$$

Ex: $\int_0^\infty \frac{\sin x}{x}\,dx$



$$f(z)=\frac{e^{iz}-1}{z}$$

$$e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots \qquad . \quad R=\frac{1}{\lim a_n^{1/n}}=+\infty$$

$$\int_{\gamma_1} f(z)\,dz = \int_\varepsilon^R f(x)\,dx = \int_\varepsilon^R \frac{\cos x-1}{x}\,dx + i\int_\varepsilon^R \frac{\sin x}{x}\,dx .$$

$$f(x) = \frac{e^{ix}-1}{x} = \frac{(\cos x+i\sin x)-1}{x} = \underbrace{\frac{\cos x-1}{x}}_{u} + i\cdot\underbrace{\frac{\sin x}{x}}_{v}$$

$\gamma_2: \theta \mapsto R\cdot e^{i\theta}, \quad 0\le\theta\le\pi.$ $\qquad Re^{i\theta}=R\cos\theta+iR\sin\theta$

$$f(z)=f(Re^{i\theta}) = \frac{e^{i(Re^{i\theta})}-1}{R\cdot e^{i\theta}} = \frac{e^{-R\sin\theta+iR\cos\theta}-1}{R\cdot e^{i\theta}}$$

$$dz=d(Re^{i\theta})=Re^{i\theta}\cdot i\,d\theta$$

$$\int_{\gamma_2} f(z)\,dz = \int_0^\pi \frac{e^{-R\sin\theta+iR\cos\theta}-1}{\cancel{Re^{i\theta}}}\cdot\cancel{Re^{i\theta}}\,i\,d\theta$$

$$= \underbrace{\left|\int_0^\pi e^{-R\sin\theta}\, e^{iR\cos\theta}\cdot i\,d\theta\right|}_{\le\ \int_0^\pi e^{-R\sin\theta}\,d\theta\ \xrightarrow{R\to+\infty}\ 0} - \underbrace{\int_0^\pi 1\cdot i\,d\theta}_{=\ i\cdot\pi} \xrightarrow{R\to+\infty} -i\cdot\pi$$

$$\cdot \int_{\gamma_3} f(z)\,dz = \int_{-R}^{-\varepsilon} \frac{e^{it}-1}{t}\,dt = \int_{-R}^{-\varepsilon} \frac{\cos t + i\sin t - 1}{t}\,dt$$

$\gamma_3 : t \to t, \ -R \le t \le -\varepsilon.$

$$\parallel$$

$$\int_{-R}^{-\varepsilon} \frac{\cos t - 1}{t}\,dt + i \int_{-R}^{-\varepsilon} \frac{\sin t}{t}\,dt$$

$$f(z) = \frac{e^{iz}-1}{z}$$

$$\int_{R}^{\varepsilon} \frac{\cos t - 1}{t}\,dt \qquad i\int_{R}^{\varepsilon} \frac{\sin x}{x}(-dx)$$

$$-\int_{\varepsilon}^{R} \frac{\cos t - 1}{t}\,dt \qquad i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx$$

$\cdot \ \gamma_4 : \theta \mapsto \underset{\overset{\parallel}{z}}{\varepsilon \cdot e^{+i\theta}}, \quad \theta : \pi \to 0.$

$dz = \varepsilon \cdot e^{i\theta} \cdot i\,d\theta \qquad \frac{dz}{z} = \varepsilon i\,d\theta$

$$f(z)\,dz = \frac{e^{iz}-1}{z}\,dz = \frac{e^{i(\varepsilon \cdot e^{i\theta})}-1}{\cancel{\varepsilon \cdot e^{i\theta}}}\cancel{\varepsilon e^{i\theta}}\,i\,d\theta = \left(e^{-\varepsilon\sin\theta} \cdot e^{i\varepsilon\cos\theta} - 1\right) i\,d\theta$$

$$e^{i \cdot \varepsilon \cdot (\cos\theta + i\sin\theta)}$$

$$\int_{\gamma_4} f(z)\,dz = \int_{\pi}^{0} \underbrace{\left(e^{-\varepsilon\sin\theta} \cdot e^{i\varepsilon\cos\theta} - 1\right)}_{\downarrow\, \varepsilon \to 0 \atop 0} i\,d\theta \xrightarrow{\varepsilon \to 0} 0$$

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \int_{\gamma_4} |f(z)|\,|dz| \le \max_{\gamma_4} |f(z)| \cdot \underset{\pi \cdot \varepsilon}{L(\gamma_4)}$$

$$f(z) = \frac{e^{iz}-1}{z} = \frac{e^{-\varepsilon\sin\theta} e^{i\varepsilon\cos\theta} - 1}{\varepsilon \cdot e^{i\theta}} \sim \frac{-\cancel{\varepsilon}\sin\theta + i\cancel{\varepsilon}\cos\theta}{\cancel{\varepsilon} \cdot e^{i\theta}}$$

$$\parallel$$

$$\frac{(\cancel{1} + iz + \frac{(iz)^2}{2!} + \cdots) - \cancel{1}}{z} = i + O(|z|).$$

$$0 = \int_\gamma f(z)\,dz = \underline{\int_{\gamma_1}} + \underline{\int_{\gamma_2}} + \underline{\int_{\gamma_3}} + \underline{\int_{\gamma_4}}$$

$$= 2i\int_\varepsilon^R \frac{\sin x}{x}\,dx + \left(\underbrace{\varphi_1(R)}_{\downarrow \atop 0} - i\pi\right) + \left(\underbrace{\varphi_2(\varepsilon)}_{\downarrow \atop 0}\right)$$

$$\xrightarrow[R\to+\infty]{\varepsilon\to 0} \; = 2i\int_0^\infty \frac{\sin x}{x}\,dx - i\pi = 0.$$

$$\Longrightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$