$z = x + iy$

$f(z)$ is complex differentiable : $\underline{f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \text{ exists}}$

$\parallel$

$u + i \cdot v = u(x,y) + i v(x,y)$.

$\Longleftrightarrow$ CR equation : $\boxed{\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}}$ $\Longleftrightarrow$ $\frac{\partial f}{\partial \bar{z}} = 0$

"$\Longrightarrow$" $+$ $\qquad \parallel$

u, v are continuously differentiable $\qquad \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) f$

$$\boxed{\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)}$$

"$\Longleftarrow$" $\quad f(z+h) - f(z) = \underline{(u(z+h) - u(z))} + i\underline{(v(z+h) - v(z))}$.

$/\!/ \; h = h_1 + i h_2$ $\qquad \parallel$

$(u(x+h_1, y+h_2) - u(x,y))$ $\qquad \underline{i\left(\frac{\partial v}{\partial x} \cdot h_1 + \frac{\partial v}{\partial y} \cdot h_2 + |h| \cdot \psi_2(h)\right)}$

$\parallel \qquad\qquad \parallel$ CR

$\frac{\partial u}{\partial x}(z) \cdot h_1 + \frac{\partial u}{\partial y}(z) \cdot h_2 + |h| \cdot \psi_1(h)$ $\qquad i\left(-\frac{\partial u}{\partial y} h_1 + \frac{\partial u}{\partial x} h_2 + |h| \cdot \psi_2(h)\right)$

$|h| = \sqrt{h_1^2 + h_2^2}$, $\psi_1(h) \downarrow 0$

$\parallel$

$$\frac{\partial u}{\partial x}(h_1 + i h_2) + \frac{\partial u}{\partial y} \cdot \underbrace{(h_2 - i h_1)}_{= -i \cdot (h_1 + i h_2)} + |h| \cdot (\psi_1(h) + i \psi_2(h))$$

$\parallel$

$$f(z+h) - f(z) = \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}\right) \cdot h + |h| \cdot (\psi_1 + i \psi_2).$$

$$\frac{f(z+h) - f(z)}{h} = \underbrace{2 \cdot \frac{1}{2}\left(\frac{\partial u}{\partial x}(z) - i \frac{\partial u}{\partial y}(z)\right)}_{= 2 \cdot \frac{\partial}{\partial z} u} + \underbrace{\frac{|h|}{h}(\psi_1 + i \psi_2)}_{\downarrow 0} \xrightarrow{h \to 0} \underbrace{2 \cdot \frac{\partial u}{\partial z} \overset{CR}{=} \frac{\partial f}{\partial z}}_{= \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)(u + iv)}$$

$\underline{f \text{ is holomorphic} \Longleftrightarrow \text{complex differentiable}} \Longleftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$.

Cor: f holomorphic $\Longrightarrow$ $\begin{cases} \Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u \overset{CR}{=} \frac{\partial}{\partial x} \cdot \frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0. \\ \Delta v = 0 \end{cases}$

$\parallel$

$u + iv$

$\{\text{Re} f, \text{Im} f$ are $\frac{(\text{conjugate})}{}$ harmonic functions$\}$

· Power series. $\sum_{n=1}^{\infty} a_n \cdot z^n = f(z)$.

$R=\infty \rightsquigarrow \underline{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}}$, $(e^z)' = \sum_{n=0}^{\infty} \frac{n \cdot z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.

$R=\infty \rightsquigarrow \underline{\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}}$, $\underline{(\cos z)' = \sum_{n=0}^{\infty} (-1)^n \cdot 2n \cdot \frac{z^{2n-1}}{(2n)!}} = -\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

$$\parallel$$

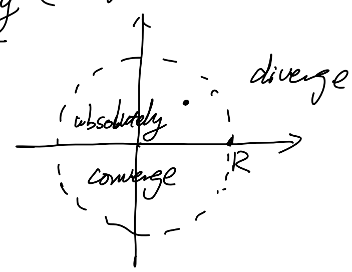
$$\underline{-(\sin z)}.$$

$S_N(z) = \sum_{n=0}^{N} a_n \cdot z^n$. $S_N(z) \xrightarrow{N \to +\infty} f(z)$.

Thm: Given a power series $\underline{\sum_{n=0}^{\infty} a_n \cdot z^n}$, there exists $0 \le R \le \infty$ s.t.

(1) If $\underline{|z| < R}$, then the series converges absolutely $\left(\sum_{n=0}^{\infty} |a_n| \cdot |z|^n \text{ converges} \right)$

(2) If $|z| > R$, the series diverges.



Hadamard's formula: $\boxed{\frac{1}{R} = \overline{\lim} \, |a_n|^{1/n}}$

Pf: $\underline{L = \frac{1}{R}}$. If $|z| < R$. $\underline{(L+\varepsilon) \cdot |z| = r < 1}$

$|a_n|^{1/n} \le L + \varepsilon$ for all $n \gg 1$. $\Rightarrow \boxed{|a_n| \cdot |z|^n \le (L+\varepsilon)^n |z|^n = r^n}$

$\Rightarrow \underline{\sum_{n=0}^{\infty} |a_n| \cdot |z|^n \le \sum_{n=0}^{\infty} r^n}$ converges. $\Rightarrow \sum_{n=0}^{\infty} a_n \cdot z^n$ converges.

$$|S_N - S_{N'}| = \left| \sum_{n=N'+1}^{N} a_n \cdot z^n \right|$$

$$\wedge$$

$$\underline{\sum_{n=N'+1}^{N} |a_n| \cdot |z^n|}$$

$|z| > R \Rightarrow |a_n| \cdot |z|^n \not\to 0 \Rightarrow$ divergence.

$\underline{\frac{1}{1-z} = 1 + z + z^2 + \cdots}$. $R = 1$ diverges on $|z| = 1$.

Thm: $f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n$ defines a holomorphic function in its disk of convergence.

$$f'(z) = \underbrace{\sum_{n=0}^{\infty} n \cdot a_n \cdot z^{n-1}}_{=\, g(z)} \quad \text{has the same radius of convergence as } f.$$

$$\sum_{n=0}^{\infty} n \cdot a_n \cdot z^{n-1} = \sum_{n=1}^{\infty} n \cdot a_n \cdot z^{n-1}$$

Pf: $\overline{\lim} (n \cdot a_n)^{\frac{1}{n}} = \underbrace{\left(\lim n^{\frac{1}{n}}\right)}_{=1} \overline{\lim} (a_n)^{\frac{1}{n}} = \overline{\lim} (a_n)^{\frac{1}{n}} = \frac{1}{R}$

$|z| < R$. $\quad \frac{f(z+h) - f(z)}{h} \xrightarrow[?]{h \to 0} g(z)$.

Set $S_N(z) = \sum_{n=0}^{N} a_n \cdot z^n$, $E_N(z) = \sum_{n=N+1}^{\infty} a_n \cdot z^n$. $\quad f(z) = S_N(z) + E_N(z)$.

Fix z_0, $|z_0| < R$. Choose $r > 0$ s.t. $|z_0| < r < R$.

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \underbrace{\left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right)}_{\mathrm{I}} + \underbrace{\left(S_N'(z_0) - g(z_0)\right)}_{\mathrm{II}} + \underbrace{\left(\frac{E_N(z_0+h) - E_N(z_0)}{h}\right)}_{\mathrm{III}}.$$

$$\mathrm{III} = \frac{E_N(z_0+h) - E_N(z_0)}{h} = \sum_{n=N+1}^{\infty} \frac{1}{h} \cdot a_n \cdot \left((z_0+h)^n - z_0^n\right) = \sum_{n=N+1}^{\infty} \frac{1}{\cancel{h}} a_n \cdot \sum_{k=0}^{n-1} (z_0+h)^k \cdot z_0^{n-1-k} \cdot \cancel{h}$$

when $|h| \ll 1$, $|z_0+h| \leq |z_0| + |h| < r$.

$\Rightarrow |\mathrm{III}| \leq \sum_{n=N+1}^{\infty} |a_n| \cdot n \cdot r^{n-1} < \varepsilon$ when $n \geq N_1 \gg 1$.

$S_N'(z_0) \xrightarrow{N \to \infty} g(z_0) \Rightarrow \exists N_2 \gg 1$ s.t. $|S_N'(z_0) - g(z_0)| < \varepsilon$ when $n \geq N_1 \gg 1$.

Finally, because $\frac{S_N(z_0+h) - S_N(z_0)}{h} \xrightarrow{h \to 0} S_N'(z_0)$, when $|h| \ll 1$, $|\mathrm{III}| < \varepsilon$.

So we get $\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = g(z_0)$. ∎

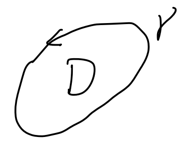
Integration along curve γ: $z = z(t)$, $a \leq t \leq b$. (piecewise differentiable)

f a continuous complex valued function.

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Thm (Cauchy Thm) If $f(z)$ is holomorphic in $U \subset \mathbb{R}^2$. $\gamma \subset D$ encloses a simply connected region $D \subset U$. then

$$\int_\gamma f(z)\,dz = 0.$$



Proof assuming that $f(z) = u(x,y) + i v(x,y)$ is continuously differentiable.

By Stoke's Thm ($\leftrightarrow$ Green's Thm)

$$\int_\gamma f(z)\,dz = \iint_D d(f(z)\,dz) = \iint_D \left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}\right)\wedge dz$$

$$\overset{dz\wedge dz = 0}{=} \iint_D \frac{\partial f}{\partial \bar{z}}\,d\bar{z}\wedge dz \;\underset{\frac{\partial f}{\partial \bar{z}} = 0}{\overset{f \text{ holomorphic}}{=}}\; 0$$

∎