Complex numbers $\mathbb{C} = \{ \underset{\substack{\| \\ z}}{a+bi} \ ; a, b \in \mathbb{R} \}$

$a = Re(z)$, $b = Im(z)$. $i^2 = -1$. $i = \sqrt{-1}$.

$(\mathbb{R}, +, \cdot)$

$\cap$

$\underline{(\mathbb{C}, +, \cdot)}$: · $z_1 + z_2 = \underline{(a_1 + i b_1)} + \underline{(a_2 + i b_2)} = (a_1 + a_2) + i(b_1 + b_2)$

is a field . · $z_1 \cdot z_2 = (a_1 + i b_1)(a_2 + i b_2) = a_1 \cdot a_2 + i a_1 b_2 + i b_1 a_2 + \underset{\substack{\| \\ -1}}{\boxed{i^2}} \underline{b_1 b_2}$

$= \underline{(a_1 a_2 - b_1 b_2)} + i \underline{(a_1 b_2 + a_2 b_1)} = z_2 \cdot z_1$

· $0 = 0 + 0 \cdot i$, $0 + z = z$, $\forall z \in \mathbb{C}$

$1 = 1 + 0 \cdot i$, $1 \cdot z = z$.

· $-z = -(a+bi) = (-a) + (-b) i$. $z + (-z) = 0$

· $z = a + bi \neq 0$ ($\Leftrightarrow$ $a \neq 0$ or $b \neq 0$).

$z^{-1} = \frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$.

$\bar{z} = \overline{a+bi} = a - bi$. $z \cdot \bar{z} = \bar{z} \cdot z = a^2 + b^2 = |z|^2$

Fact:

$\underline{\mathbb{C} \text{ is algebraically closed field}}$: Every polynomial with $\mathbb{C}$-coefficients $\underline{\text{always have a } \overset{\substack{\mathbb{C} \\ \Downarrow}}{\text{root}}}$.

$\Updownarrow$

always split.

$p(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$

\# roots = degree of polynomial

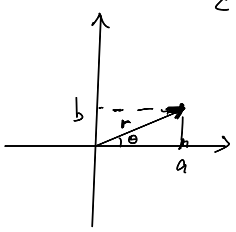
$\exists \ w \in \mathbb{C}$, s.t. $p(w) = 0$.

For $\mathbb{R}$, $x^2 + 1 = 0$ no root in $\mathbb{R}$.

$$z=a+bi = r\cdot\cos\theta+i\cdot r\sin\theta = r\cdot\underline{(\cos\theta+i\cdot\sin\theta)}$$

$$\parallel$$

$$r\cdot e^{i\theta}$$



Euler's formula: $e^{i\theta}=\cos\theta+i\sin\theta$

$(e^{i\pi}=-1)$

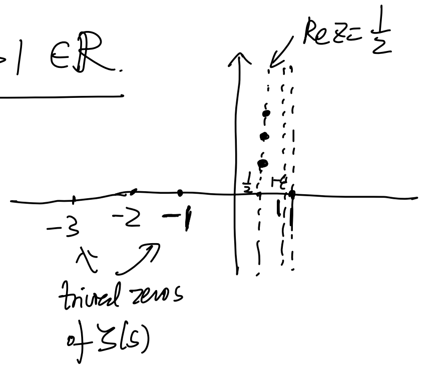
$$\underline{e^{x}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$$

$i^2=-1,\ i^3=-i,\ i^4=(-1)^2=1$

$$\underline{e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}}+\cdots=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)$$

$$=\cos\theta+i\cdot\sin\theta$$

$$\underline{\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\ \text{ converges when }\ s>1\in\mathbb{R}.}$$



$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

$$\zeta(2k)=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=c\cdot \underset{\uparrow}{B_k}$$

Bernoulli numbers.

ζ(s) extends to a holomorphic fct. on $\mathbb{C}\setminus\{1\}$.
(analytic)

$$\zeta(-1)=1+2+3+4+\cdots=\textcircled{$-\frac{1}{12}$}\quad(\text{Euler})$$

Prime Number Thm.

$$\underline{\pi(x)\sim\frac{x}{\log x}}$$

$$\parallel$$

$$\#\{p:\text{prime},\ p\le x\}$$

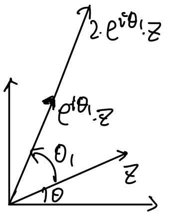
$$\Updownarrow$$

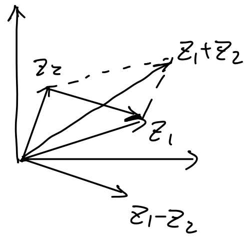
$$\boxed{\zeta \text{ has no zeros on } \mathrm{Re}\, z=1.}$$

$z = r\cdot e^{i\theta}$. $z_1\cdot z_2 = r_1\cdot e^{i\theta_1}\cdot r_2\cdot e^{i\theta_2} = (r_1 r_2)\cdot e^{i(\theta_1+\theta_2)}$

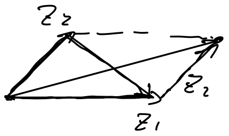
$\underline{e^{i\theta_1}}\cdot z = e^{i\theta_1}\cdot r e^{i\theta} = r\cdot e^{i(\theta+\theta_1)}$.

$\theta = \mathrm{Arg}(z)$
argument of z





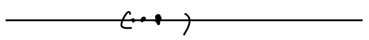
$z = a+bi = (a,b)$.



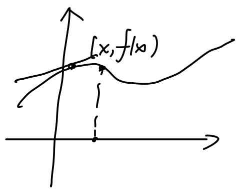
$|z| = |a+ib| = \sqrt{a^2+b^2}$ $|z_1 \pm z_2| \leq |z_1| + |z_2|$ triangle inequality

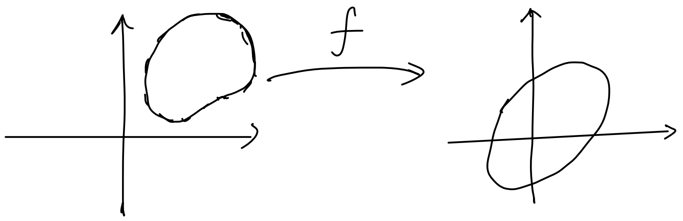
$\{z_n\}_{n\geq 1}$ sequence of complex numbers.

$\lim\limits_{n\to+\infty} z_n = w$ if $\lim\limits_{n\to+\infty} \underline{|z_n - w|} = 0$.



- Complex valued function: $f: \underset{\mathbb{C}}{\underset{\cap}{D}} \longrightarrow \mathbb{C}$





- Continuous at $z_0 \in D$: $\displaystyle \lim_{\substack{z \to z_0 \\ z \in D}} f(z) = f(z_0)$

- $\underline{\text{complex differentiable}}$ at $z_0 \in D$: $\displaystyle \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists.

$$\underset{\text{analytic}}{\overset{\Updownarrow}{\underset{\Updownarrow}{\text{holomorphic}}}} \qquad\qquad \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} \overset{\shortparallel}{=} f'(z_0).$$

f is - - - - - in an open domain D: f is complex differentiable at any point $z \in D$.

$\Longrightarrow$. f is infinitely many $\underset{\text{times}}{\text{ }}$ differentiable. (i.e. $f', f'', \ldots, f^{(k)}, \ldots$) exist for any k

- $\underset{\text{(analytic)}}{}$ $f(z) = \underbrace{\sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n}_{\text{absolutely convergent}}$ is true in a neighborhood of $z_0 \in D$.

Assume
u, v are differentiable
on (x,y)

- $\underset{\overset{\shortparallel}{x+iy}}{f(z)} = u(z) + i \cdot v(z) = \underline{u(x,y)} + i \cdot \underline{v(x,y)}$.

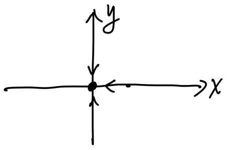
$$\underline{\lim_{\substack{h\to 0 \\ \shortparallel \\ s+it}} \frac{f(z+h)-f(z)}{h}}.$$

$$\lim_{s\to 0} \frac{f(z+\overset{\overset{s+i\cdot 0}{\shortparallel}}{s})-f(z)}{s} = \lim_{s\to 0} \frac{u(x+s,y)+i\,v(x+s,y)-u(x,y)-i\,v(x,y)}{s}$$

$z=(x,y)$, $h=(s,0)$

$$= \lim_{s\to 0} \frac{u(x+s,y)-u(x,y)}{s} + i \cdot \lim_{s\to 0} \frac{v(x+s,y)-v(x,y)}{s}$$

$$= \underline{\frac{\partial u}{\partial x}(x,y) + i \cdot \frac{\partial v}{\partial x}(x,y)}$$



- $\displaystyle \lim_{t\to 0} \frac{f(z+it)-f(z)}{it} = \lim_{t\to 0} \frac{u(x,y+t) + \boxed{i}\,v(x,y+t) - u(x,y) - i\,v(x,y)}{\boxed{i}\,t}$. $\quad \frac{1}{i} = -i$

$h = it = (0,t)$

$$= \lim_{t\to 0} \frac{v(x,y+t)-v(x,y)}{t} - i \cdot \lim_{t\to 0} \frac{u(x,y+t)-u(x,y)}{t} = \underline{\frac{\partial v}{\partial y}(x,y) - i \frac{\partial u}{\partial y}}.$$

f is differentiable $\Longrightarrow$ $\underline{\frac{\partial u}{\partial x}} + i\,\underline{\frac{\partial v}{\partial x}} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

$$\Updownarrow$$

$$\boxed{\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}}$$

(Cauchy-Riemann equation)
CR

Thm: f is differentiable $\Longleftrightarrow$ CR eq. are satisfied.

$$\begin{cases} z = x + i\cdot y \\ \bar{z} = x - iy \end{cases} \Longleftrightarrow \begin{cases} x = \frac{z+\bar{z}}{2} \\ y = \frac{z-\bar{z}}{2i} \end{cases}.$$

$$f(z) = \underline{f(x,y)} = f\left(\frac{z+\bar z}{2}, \frac{z-\bar z}{2i}\right)$$
$$\overset{\parallel}{f(z,\bar z)}.$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$

$$\begin{tikzcd} \enclose{circle}{z} \arrow[r, dash] \arrow[dr, dash] & x \arrow[dr, dash] & \\ & & f \\ \bar z \arrow[r, dash] \arrow[ur, dash] & y \arrow[ur, dash] & \end{tikzcd}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \qquad \enclose{roundedbox}{\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)}$$

"f does not depend on $\bar z$"

$$\Updownarrow$$

$$\underset{\substack{\parallel \\ u+iv}}{f} \text{ differentiable} \Longleftrightarrow \begin{cases} 0 = \underline{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}} = \operatorname{Re}\left(\frac{\partial f}{\partial \bar z}\right) \\ 0 = -\frac{u}{y} + \frac{\partial v}{\partial x} = \operatorname{Im}\left(\frac{\partial f}{\partial \bar z}\right) \end{cases} \Longleftrightarrow \boxed{\frac{\partial f}{\partial \bar z} = 0.}$$

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\underline{\left(\enclose{circle}{\frac{\partial}{\partial x}} + i\frac{\partial}{\partial y}\right)}\,\underline{(u+iv)} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)$$
$$\parallel$$
$$\underline{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$z \mapsto |z|^2 = z\cdot\bar z$, $z \to \bar z\cdot z^2$

$z \mapsto z^2$