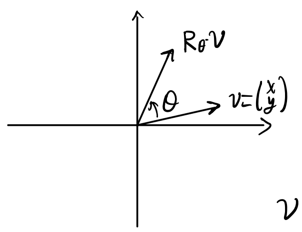
- Rotations on $\mathbb{R}^2$ by using multiplication of complex numbers:



$$R_\theta \cdot v = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta \cdot x - \sin\theta \cdot y \\ \sin\theta \cdot x + \cos\theta \cdot y \end{pmatrix}$$

$$v \Leftrightarrow z = x + iy$$

$$R_\theta \cdot v \Leftrightarrow e^{i\theta} \cdot z = (\cos\theta + i\sin\theta)(x+iy)$$
$$= (\cos\theta \cdot x - \sin\theta \cdot y) + i \cdot (\sin\theta \cdot x + \cos\theta \cdot y)$$

- Rotations of S^2 by using multiplication of quaternions:

$$S^2 = \{ xi + yj + zk : x, y, z \in \mathbb{R},\ x^2+y^2+z^2=1 \}$$
$$= \{ p \in \mathbb{H} : |p|^2 = 1,\ \bar{p} = -p \}$$

$$S^3 = \{ a + bi + cj + dk : a^2+b^2+c^2+d^2 = 1 \}$$
$$= \{ q \in \mathbb{H} : |q|^2 = 1 \}$$

$$S^3 \longrightarrow \mathrm{Isom}^+(S^2)$$
$$q \longmapsto f_q : S^2 \to S^2 \quad f_q(p) = q \cdot p \cdot q^{-1} \quad \forall p \in S^2.$$

Any $q \in S^3$ can be represented as:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk) \quad \text{with} \quad l^2 + m^2 + n^2 = 1.$$

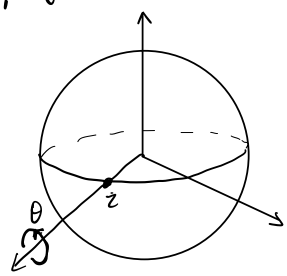
$$|q|^2 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}(l^2+m^2+n^2)$$
$$= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

Thm: f_q represents the rotation around the axis passing through 0 and $(l,m,n)\in S^2$ and with angle θ. In particular, any rotation of S^2 can be represented as f_q with $q\in S^3$.

Proof: 1. We first prove that when $q=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}i$, f_q is the rotation around the x-axis (passing through 0 and $(1,0,0)$) with angle θ.

Choose any $p = xi+yj+zk \in S^2$. We calculate $f(p)=q\cdot p\cdot q^{-1}$.

$$p = xi+yj+zk = xi+(y+zi)j = xi+wj \quad \text{with } w=y+zi$$



$$|q|^2 = q\cdot\bar{q} = 1 \Rightarrow q^{-1} = \bar{q} = \cos\frac{\theta}{2}-\sin\frac{\theta}{2}i.$$

$$q\cdot p\cdot q^{-1} = \left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}i\right)\left(xi+(y+iz)j\right)\cdot\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}i\right)$$

$$= e^{i\frac{\theta}{2}}\cdot xi\cdot e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}\cdot wj\cdot e^{-i\frac{\theta}{2}}$$

$$= e^{i\frac{\theta}{2}}\cdot e^{-i\frac{\theta}{2}}\,xi + e^{i\frac{\theta}{2}}w\cdot\overline{e^{-i\frac{\theta}{2}}}\cdot j$$

$$= xi + e^{i\theta}\cdot w\cdot j = xi+e^{i\theta}(y+zi)\cdot j$$

We use the properties: multiplication of complex numbers is commutative.

$$j\cdot(c+di) = cj-dk = (c-di)j \Leftrightarrow j\cdot w = \bar{w}j$$

So $f_q(p)$ is obtained from p by rotating the (y,z)-plane with the angle θ as wanted.

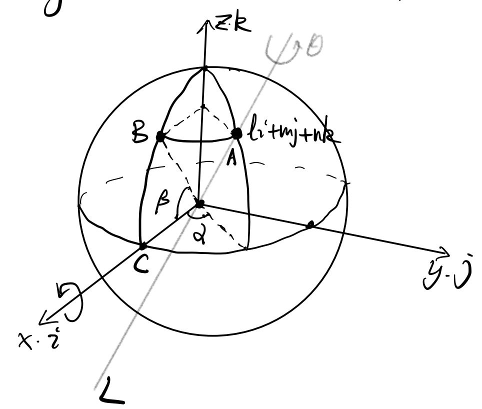
Similarly, one can show that:

$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} j \longrightarrow f_q =$ rotation around y-axis by angle θ

$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} k \longrightarrow f_q =$ rotation around z-axis by angle θ.

2. Let $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk)$ be general element in S^3.

We can realize the rotation around the axis passing through O and $li+mj+nk$ with angle θ by composition of 3 steps:



First rotate the axis L to the x-axis:

$$P \mapsto f_1(P) = \underbrace{q_2(\beta)\, \underbrace{q_3(-\alpha) \cdot P \cdot q_3(-\alpha)^{-1}}_{A \to B}\, q_2(\beta)^{-1}}_{B \to C}$$

where $q_3(-\alpha) = \cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} k$

$q_2(\beta) = \cos\frac{\beta}{2} + \sin\frac{\beta}{2} j$

Second rotate around x-axis by angle θ:

$$f_1(P) \longmapsto f_2 \circ f_1(P) = q_1(\theta) f_1(P) \cdot q_1(\theta)^{-1}$$ where $q_1(\theta) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} \cdot i$

Third rotate the x-axis back to the axis $li+mj+nk$:

$$f_2 \circ f_1(P) \longmapsto f_3 \circ f_2 \circ f_1(P) = q_3(\alpha) q_2(-\beta) \cdot f_2 \circ f_1(P)\, q_2(-\beta)^{-1} \cdot q_3(\alpha)^{-1}$$

Altogether:

$$p \longmapsto f_3 \circ f_2 \circ f_1(p)$$

$$\|$$

$$\underbrace{q_3(\alpha)\, q_2(-\beta) \cdot q_1(\theta)\, q_2(\beta)\, q_3(-\alpha)}_{Q} \cdot p\, \underbrace{q_3(-\alpha)^{-1} q_2(\beta)^{-1} \cdot q_1(\theta)^{-1}\, q_2(-\beta)^{-1}\, q_3(\alpha)^{-1}}_{Q^{-1}}$$

$q_2(-\beta) = \overline{q_2(\beta)} = q_2(\beta)^{-1}$

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$$Q = q_3(\alpha)\, q_2(-\beta) \cdot q_1(\theta)\, q_2(\beta) \cdot q_3(-\alpha)$$

$$= q_3(\alpha)\, q_2(-\beta) \cdot \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2} \cdot i\right) q_2(\beta) \cdot q_3(-\alpha)$$

$$= \cos\frac{\theta}{2} \cdot q_3(\alpha) \cdot \overline{q_2(\beta)} \cdot q_2(\beta) \cdot \overline{q_3(\alpha)} + \sin\frac{\theta}{2} \cdot q_3(\alpha) \cdot q_2(-\beta)\, i \cdot q_2(\beta) \cdot q_3(-\alpha)$$

$$= \cos\frac{\theta}{2} \, |q_2(\beta)|^2 |q_3(\alpha)|^2 + \sin\frac{\theta}{2} \cdot \underbrace{q_3(\alpha) \cdot \underbrace{q_2(-\beta)\, i \cdot q_2(-\beta)^{-1}}_{i: A \to B} \cdot q_3(\alpha)^{-1}}_{A \to B \to C}$$

$$= \cos\frac{\theta}{2} + \sin\frac{\theta}{2} \cdot (li + mj + nk) = q$$ that we started with.

This proves the theorem: When $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk)$,

$f_q(p) = q \cdot p \cdot q^{-1}$ represents the rotation with axis $li + mj + nk$ and angle θ.

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