Group of isometries of the Euclidean plane:

$\mathrm{Isom}(\mathbb{R}^2) = \{$composition of (at most) 3 reflections$\}$

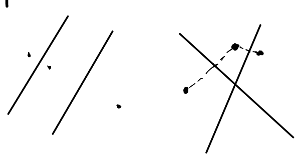
$= \{$translations, rotations, glide reflections$\}$

Subgroup of orientation preserving isometries:

$\mathrm{Isom}^+(\mathbb{R}^2) = \{$composition of even number of reflections$\}$

$= \{$composition of 2 reflections$\}$

$= \{$translation, rotations$\}$

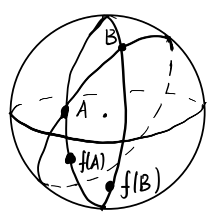


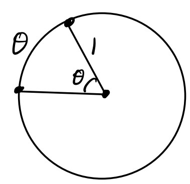
Isometry group of S^2:

$\mathrm{Isom}(S^2) = \{ f: S^2 \to S^2,\ \mathrm{dist}(f(A), f(B)) = \mathrm{dist}(A,B) \}$

$= \{$composition of at most 3 reflections$\}$ across great circles.

reflection across the equator:





$\mathrm{Isom}^+(S^2) = \{$orientation preserving isometries of $S^2\}$

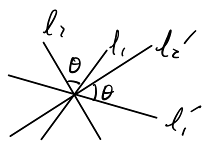
$= \{$rotations$\}$

$= \{$composition of 2 reflections$\}$

__Lemma:__ Composition of rotations of S^2 is also a rotation.

Pf:

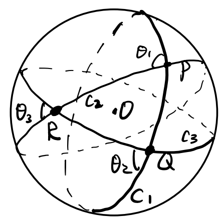
On plane:



$r_{l_2} \circ r_{l_1} = r_{l_2'} \circ r_{l_1'} =$ rotation by 2θ

$\Rightarrow$ one mirror can be any line passing through the center and then the other mirror is determined by it and the angle.

Same statement is true for rotations of S^2.



rotation around axis OP by angle $2\theta_1$:

$$f = r_{C_2} \circ r_{C_1}$$

rotation around the axis OQ by angle $2\theta_2$:

$$g = r_{C_1} \circ r_{C_3}$$

$\Rightarrow$ $f \circ g = r_{C_2} \circ r_{C_1} \circ r_{C_1} \circ r_{C_3} = r_{C_2} \circ r_{C_3}$ is the rotation around the axis OR by the angle $2\theta_3$ $\blacksquare$

- Use quaternions to represent rotations of S^2.

<u>Quaternions</u>: $\mathbb{H} = \{a + bi + cj + d\cdot k \,;\, a, b, c, d \in \mathbb{R}\} \cong \mathbb{R}^4$

$$a+bi+cj+dk = (a+bi) + (c+di)j, \qquad \mathbb{R}^4 \cong \mathbb{C}^2$$

i, j, k satisfies: $i^2 = j^2 = k^2 = -1$,

$$i\cdot j = k = -j\cdot i, \quad j\cdot k = i = -k\cdot j, \quad k\cdot i = j = -i\cdot k.$$

multiplication: $q = a + bi + cj + dk$, $q' = a' + b'i + c'j + d'k$

$$q\cdot q' = (aa' - bb' - cc' - dd') + i\cdot(ab' + ba' + cd' - dc') + j\cdot(ac' - bd' + c\cdot a' + db') + k\cdot(ad' + bc' - cb' + da')$$

Representation of quaternions by 2x2 complex matrices:

$$1 = M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \underbrace{i = M_i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \; j = M_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; k = M_k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}_{\text{Pauli matrices.}}$$

$$q = a+bi+cj+dk \;=\; M_q = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \qquad \begin{matrix} z=a+bi \\ w=c+di \end{matrix}$$

$$M_{q_1+q_2} = M_{q_1} + M_{q_2}\,, \quad M_{q_1 q_2} = M_{q_1}\cdot M_{q_2}.$$

$\Rightarrow$ multiplication in $\mathbb{H}$ is not commutative in general but is associative:

$$\underset{\underset{(M_{q_1}\cdot M_{q_2})\cdot M_{q_3}}{\shortparallel}}{(q_1 \cdot q_2)\cdot q_3} = \underset{\underset{M_{q_1}\cdot (M_{q_2}\cdot M_{q_3})}{\shortparallel}}{q_1 \cdot (q_2 \cdot q_3)}$$

Conjugation of $q = a+bi+cj+dk$:

$$\bar{q} = a-bi-cj-dk = M_{\bar{q}} = \begin{pmatrix} a-bi & -c-di \\ c-di & a+bi \end{pmatrix} \qquad \boxed{M_{\bar{q}} = \bar{M}_q^{\,t}}$$

$$= \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix} = \begin{pmatrix} \bar{z} & \bar{w} \\ -w & z \end{pmatrix}^t = \overline{M_q}^{\,t} = M_q^* \quad \begin{pmatrix} \text{conjugated} \\ \text{transpose} \end{pmatrix}$$

- $\overline{q_1 q_2} = (M_{q_1 q_2})^* = (M_{q_1}\cdot M_{q_2})^* = M_{q_2}^* \cdot M_{q_1}^* = \bar{q}_2 \cdot \bar{q}_1$
- $1 = q\cdot q^{-1} = \overline{q \cdot q^{-1}} = \overline{q^{-1}}\cdot \bar{q} \Rightarrow \overline{q^{-1}} = \bar{q}^{-1}$

2 properties of conjugation.

Norm of q: $\quad |q|^2 = a^2+b^2+c^2+d^2 = |z|^2+|w|^2 = \det(M_q)$

$$= q\cdot\bar{q} = \bar{q}\cdot q.$$

$$q\cdot\bar{q} = M_q\cdot M_{\bar{q}} = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}\begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix} = \begin{pmatrix} |z|^2+|w|^2 & 0 \\ 0 & |w|^2+|z|^2 \end{pmatrix} = M_{|q|^2} = |q|^2.$$

$$\bar{q}\cdot q = M_{\bar{q}}\cdot M_q = \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix}\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \;/\!/$$

$$|q_1\cdot q_2|^2 = (q_1\cdot q_2)\overline{(q_1\cdot q_2)} = q_1 q_2 \bar{q}_2 \cdot \bar{q}_1 = q_1\cdot |q_2|^2 \cdot \bar{q}_1 = q_1\cdot\bar{q}_1\,|q_2|^2 = |q_1|^2\cdot|q_2|^2.$$

$$S^2 = \{(b,c,d) \in \mathbb{R}^3 ; b^2+c^2+d^2=1\}$$
$$= \{p = bi+cj+dk \in \mathbb{H} : |p|^2 = b^2+c^2+d^2=1\}$$
$$= \{p \in \mathbb{H} : \underset{\overline{p}=-p}{\underset{\Updownarrow}{Re(p)=0}},\ \underset{\det M_p}{\underset{\shortparallel}{|p|^2}}=1\}$$

$Re(a+bi+cj+dk) = a \in \mathbb{R}$.

$$S^3 = \{(a,b,c,d) \in \mathbb{R}^4 : a^2+b^2+c^2+d^2=1\}$$
$$= \{q \in \mathbb{H} : \underset{q\cdot\overline{q}}{\underset{\shortparallel}{|q|^2}} = 1\}$$

quaternions with unit norm.

$$= \{q : M_q \cdot M_q^* = I_2 ,\ \det M_q = 1\} = SU(2)$$

Any $q \in S^3$ determines a rotation of S^2:

$$S^3 \longrightarrow Isom^+(S^2)$$
$$q \longmapsto f_q : S^2 \to S^2$$
$$f_q(p) = q \cdot p \cdot q^{-1}.$$

Verify that $f_q(p) \in S^2$ for any $p \in S^2$:

$$|f_q(p)|^2 = \det M_{f_q(p)} = \det(M_{q\cdot p\cdot q^{-1}}) = \det(M_q \cdot M_p \cdot M_q^{-1})$$
$$= \det(M_q) \cdot \det(M_p) \cdot \det(M_q)^{-1} = \det(M_p) = 1$$

$q\cdot\overline{q} = 1 \Rightarrow \overline{q}^{-1} = q$
$\overline{q} = q^{-1}$

$$\overline{f_q(p)} = \overline{q\cdot p\cdot q^{-1}} = \overline{q^{-1}} \cdot \overline{p} \cdot \overline{q} = \overline{q}^{-1} \cdot \overline{p} \cdot \overline{q}$$
$$= q \cdot \overline{p} \cdot q^{-1} = -q\cdot p\cdot q^{-1} = -f_q(p).$$

so $f_q(p) \in S^2$.

Verify that f_q preserves the distance:

$$|f_q(P_2)-f_q(P_1)|^2 = |f_q(P_2-P_1)|^2 = |q|^2\cdot|P_2-P_1|^2\cdot|q^{-1}|^2$$

$$= |q|^2\cdot|P_2-P_1|^2\cdot|q|^{-2} = |P_2-P_1|^2 .$$

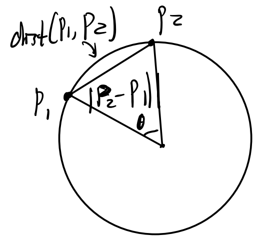
$$[0,\pi]$$

$$\Downarrow$$

$$2\cdot\sin^{-1}\left(\frac{|P_2-P_1|}{2}\right)$$

$$\shortparallel$$

$\Rightarrow f$ preserves the chordal distance



$\text{dist}(P_1,P_2)=\theta$

$|P_2-P_1|=2\cdot\sin\frac{\theta}{2}$

$\Rightarrow f$ preserves the spherical distance