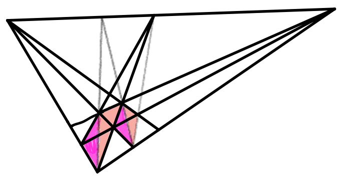
Incidence Relation:



$\Leftarrow$ projective Desargues

Group of transformations: X a topological space

$G = \{f: X \to X$ transformations satisfying certain properties$\}$

is a group if the following 3 conditions hold true:

(1) $f, g \in G \Rightarrow g \circ f \in G$

(2) $e = Id_X \in G$

(3) $\forall f \in G, \exists f^{-1} \in G$ s.t. $f \circ f^{-1} = f^{-1} \circ f = Id_X$.

$\{$translations, rotations, glide reflections$\}$

$\|$

Ex: $Isom(\mathbb{R}^2) = \{$isometries of $\mathbb{R}^2\} = \{$compositions of reflections on $\mathbb{R}^2\}$

$\cup$ $\cup$

$Isom^+(\mathbb{R}^2) = \{$orientation preserving isometries$\}$

$\|$

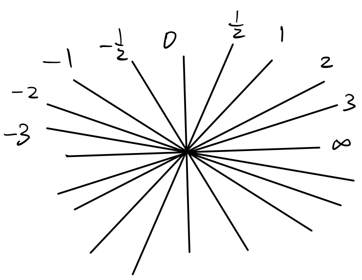
$\{$composition of even numbers of reflections$\}$

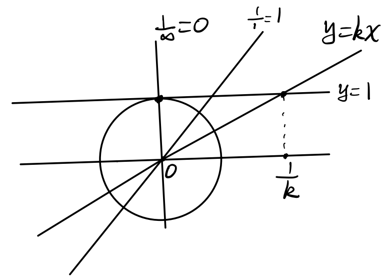
Ex. $PGL(2) = \{\text{linear fractional transformations on } \mathbb{RP}^1\}$

$$\cong \left\{ \underset{A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} : ad-bc \neq 0 \right\} \Big/ \sim$$

$A \sim kA$ for any $k \neq 0$.

$$f(x) = \frac{ax+b}{cx+d} = \frac{(ka)\cdot x + (kb)}{(kc)\cdot x + (kd)} \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$





$y=kx \rightsquigarrow$ label, $\frac{1}{k} = \frac{1}{\text{slope}}$

$y = k\cdot x \longrightarrow$ direction $(1,k)$

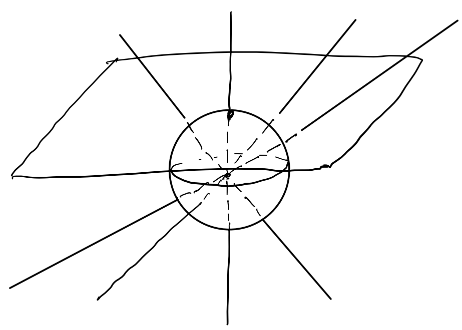
$$\rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} a+bk \\ c+dk \end{pmatrix}$$

$$\rightsquigarrow \text{label}: \frac{a+bk}{c+dk} = \frac{a\cdot\frac{1}{k}+b}{c\cdot\frac{1}{k}+d} = f\left(\tfrac{1}{k}\right)$$

$\rightsquigarrow PGL(3,\mathbb{R}) = \{3\times 3 \text{ invertible matrix } A\} / A \sim kA,\ k \neq 0$

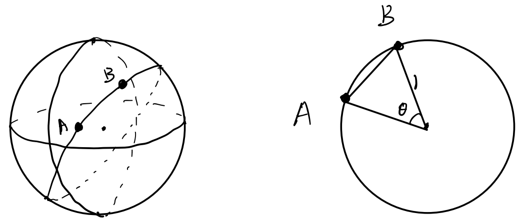
$\mathbb{RP}^2 = \mathbb{R}^2 \cup \mathbb{RP}^1$

$\cong$ 



Isometry of sphere $\quad Isom(S^2) \quad S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$

"lines" on sphere = great circles = intersection of planes passing through $O \in \mathbb{R}^3$ with the sphere.



$\text{dist}(A,B) = \theta = 2 \cdot \arcsin^{-1} \frac{|AB|}{2}$.

$|AB| = 2 \cdot \sin\frac{\theta}{2}$

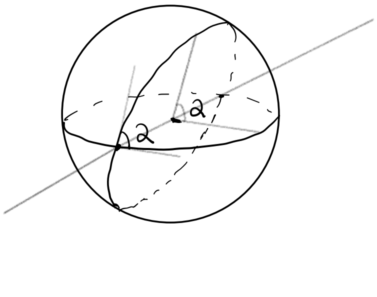
Reflections across a great circle = reflections across the corresponding plane (passing through $O \in \mathbb{R}^3$)

Three reflections Thm: Any isometry of S^2 is a composition of at most three reflections.

The argument is the same as the one for $Isom(\mathbb{R}^2)$, and is based on:

Fourth point determination: Fix any 3 points $A, B, C \in S^2$ which are not on a great circle. Then any point $D \in S^2$ is uniquely determined by ~~its~~ distance to A, B, C.

The composition of 2 reflections on S^2 is always a rotation:
(on $\mathbb{R}^2$, it can be a translation (parallel mirrors) or a rotation)



α: the angle between the mirror planes

$\Rightarrow$ rotation by angle $\theta = 2\alpha$