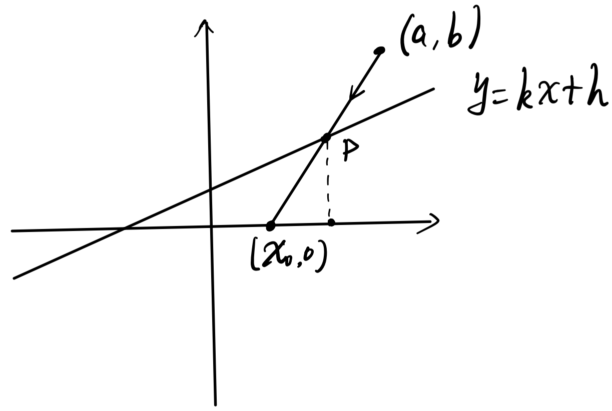
Projection



$$\begin{cases} x(t) = (1-t)a + t\cdot x_0 \\ y(t) = (1-t)b + t\cdot 0 = (1-t)b. \end{cases}$$ the line connecting (a,b) and $(x_0, 0)$

P satisfies: $(1-t)b = k((1-t)\cdot a + t\cdot x_0) + h$

$$\Rightarrow ((ka-b) + k\cdot x_0)t = -(ka+h) + b$$

$$\Rightarrow t = \frac{(-ka+b)-h}{(ka-b)+k\cdot x_0}$$

$$\Rightarrow x(t) = (1-t)\cdot a + t\cdot x_0 = \frac{Ax_0+B}{Cx_0+D}$$ for some $A, B, C, D \in \mathbb{R}$

Fractional linear transformation:

$$f(x) = \frac{Ax+B}{Cx+D} = \frac{A(x+\frac{D}{C}) + B - \frac{AD}{C}}{Cx+D} = \frac{A}{C} + \frac{BC-AD}{C^2\cdot(x+\frac{D}{C})}$$

$$= f_4 \circ f_3 \circ f_2 \circ f_1(x)$$

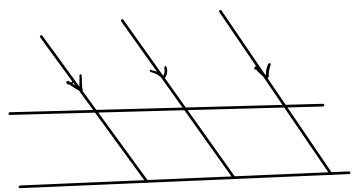
$f_1(x) = x + \frac{D}{C}$ translation

$f_2(x) = \frac{1}{x}$ inversion

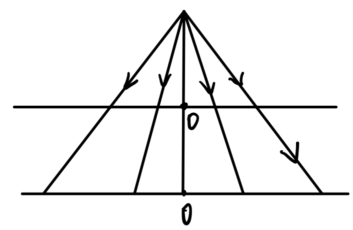
$f_3(x) = \frac{BC-AD}{C^2}\cdot x$ scaling

$$f_4(x) = \frac{A}{c} + x \quad \text{translation.}$$

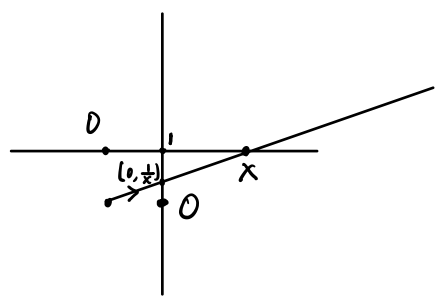
These basic operations are realized by special projections:



translation



scaling



inversion.

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cross ratio: $[p, q, r, s] = \frac{r-p}{r-q} \cdot \frac{s-q}{s-p}$.

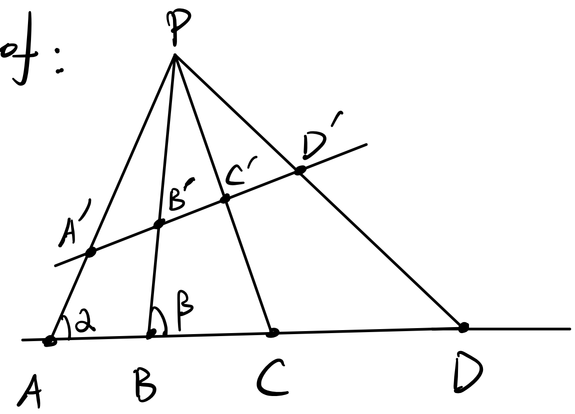
invariant under projection:

invariant under translation: $[p+c, q+c, r+c, s+c] = [p, q, r, s]$

scaling: $[k \cdot p, k \cdot q, k \cdot r, k \cdot s] = [p, q, r, s]$

inversion: $[\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}] = [p, q, r, s]$

More geometric proof:



$$\frac{|CA|}{|CB|} \cdot \frac{|DB|}{|DA|} = \frac{|C'A'|}{|C'B'|} \cdot \frac{|D'B'|}{|D'A'|}$$

Law of Sine: $\dfrac{|CA|}{\sin\angle APC} = \dfrac{|CP|}{\sin\alpha}$, $\dfrac{|CB|}{\sin\angle BPC} = \dfrac{|CP|}{\sin\beta}$

$\dfrac{|DA|}{\sin\angle APD} = \dfrac{|DP|}{\sin\alpha}$, $\dfrac{|DB|}{\sin\angle BPD} = \dfrac{|DP|}{\sin\beta}$

$$\Rightarrow \quad \frac{|CA|}{|CB|} \cdot \frac{|DB|}{|DA|} = \frac{\sin\angle APC}{\sin\angle BPC} \cdot \frac{\sin\angle BPD}{\sin\angle APD} = \frac{|C'A'|}{|C'B'|} \cdot \frac{|D'B'|}{|D'A'|}.$$

<u>Fourth point determination:</u> Given any $p, q, r \in \mathbb{RP}^1 = \mathbb{R}^1 \cup \{\infty\}$.

any point $x \in \mathbb{RP}^1$ is uniquely determined by its cross-ratio.

In fact, $x \mapsto [p, q; r, x] = \frac{r-p}{r-q} \cdot \frac{x-q}{x-p}$ is a

bijective map from $\mathbb{RP}^1$ to $\mathbb{RP}^1$.

(a linear fraction transformation).

<u>Ex</u>: $[0, 1; \infty, x] = \frac{\infty - 0}{\infty - 1} \cdot \frac{x-1}{x-0} = \frac{x-1}{x}$.

$\frac{x-1}{x} = y \iff x - 1 = x \cdot y \iff x = \frac{1}{1-y}$.

$(1-y)x = 1$

<u>Existence of three-point maps and Uniqueness</u> $\forall$ three points $p, q, r \in \mathbb{RP}^1$ and three points $p', q', r' \in \mathbb{RP}^1$.

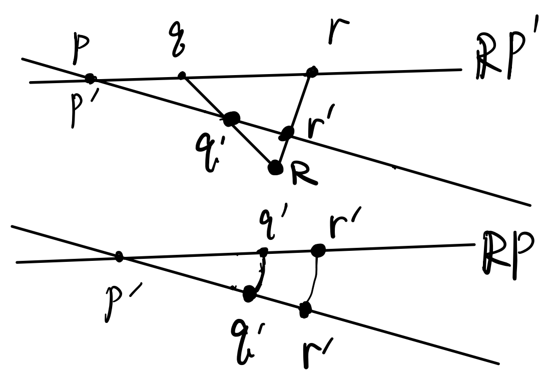
$\exists$ a unique linear fractional transformation $f : \mathbb{RP}^1 \to \mathbb{RP}^1$

s.t. $f(p) = p'$, $f(q) = q'$, $f(r) = r'$.

($f(x) = \frac{Ax+B}{Cx+D}$ satisfies $\frac{Ap+B}{Cp+D} = p'$, $\frac{Aq+B}{Cq+D} = q'$, $\frac{Ar+B}{Cr+D} = r'$)

and (A, B, C, D) determined up to multiplication by a nonzero)

Pf: Existence: linear fractional transformation is geometrically represented by a projection.



Uniqueness: linear fractional transformation preserves the cross ratio

$\Rightarrow$ $[f(p), f(q); f(r), f(x)] = [p, q; r, x]$

$\Rightarrow$ $f(x)$ is uniquely determined by $[p, q; r, x]$ ■

Fourth pt. determination

Thm: The map $f: \mathbb{RP}^1 \to \mathbb{RP}^1$ is a linear fractional transformation if and only if f preserves the cross-ratio.

Pf: Just need to prove the "if" direction. Pick any 3 pts. $p, q, r \in \mathbb{RP}^1$

By the existence of 3-pt. maps, there exists a linear fractional transformation $\tilde{f}$ s.t. $\tilde{f}(p) = f(p)$, $\tilde{f}(q) = f(q)$, $\tilde{f}(r) = f(r)$.

(we can assume $Im(f)$ contains at least 3 pts).

Then f preserves the cross-ratio $\Rightarrow \forall x \in \mathbb{R}P^1$,

$$[f(p), f(q), f(r), f(x)] = [p, q, r, x] = [\tilde{f}(p), \tilde{f}(q), \tilde{f}(r), \tilde{f}(x)]$$
$$\| $$
$$[f(p), f(q), f(r), \tilde{f}(x)]$$

Fourth pt. determination

$\Longrightarrow \quad f(x) = \tilde{f}(x)$ is a linear fractional transformation.

(Symmetry groups)

Klein: Geometry is the study of transformation groups of spaces together with their preserved quantities:

projective Geometry: {linear fractional transformations}

preserved quantity: Cross ratio

Euclid Geometry: $Isom(\mathbb{R}^2) = \{$ plane isometries $\}$

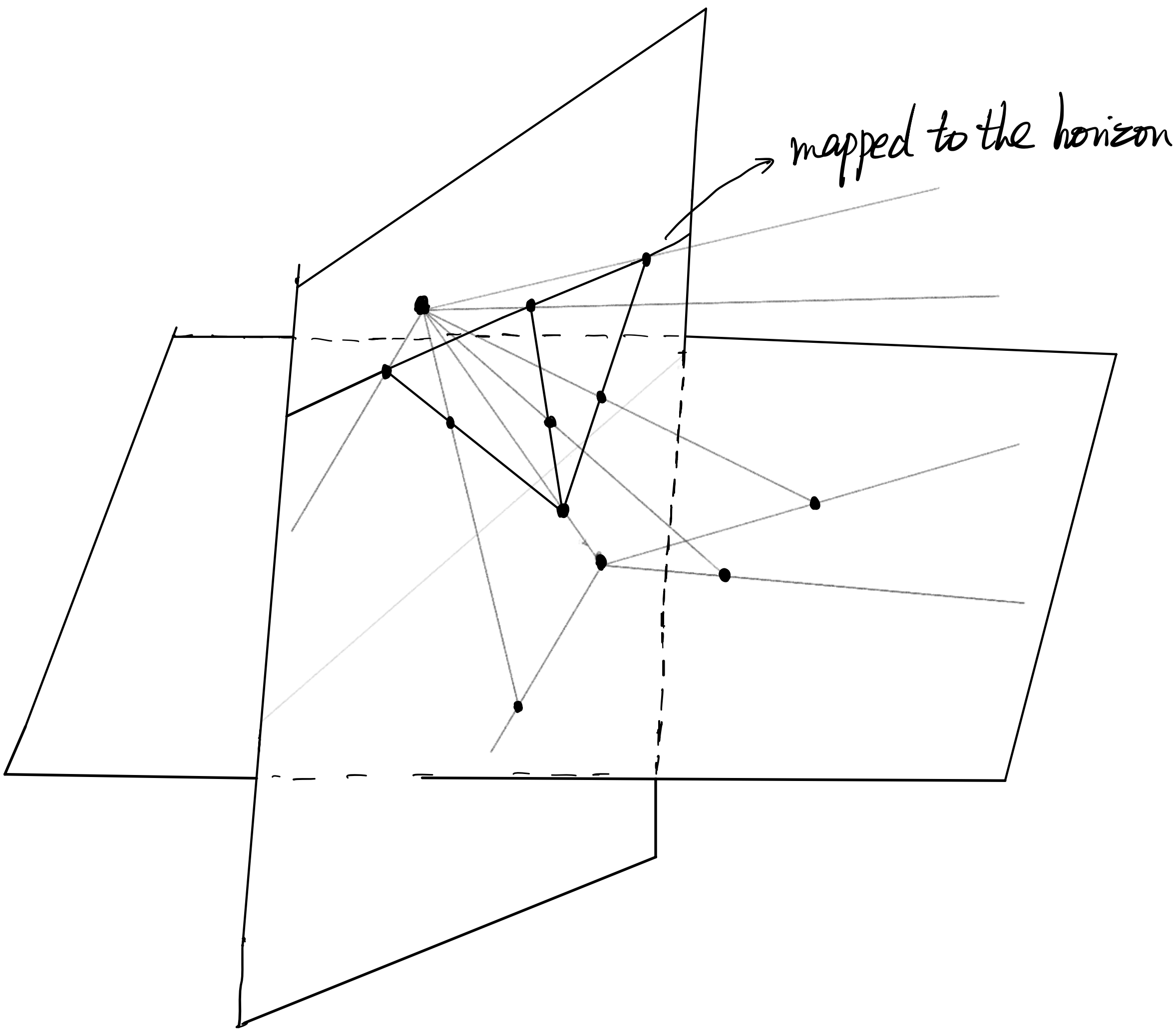
preserved quantity: distance, angles.

(Compare:

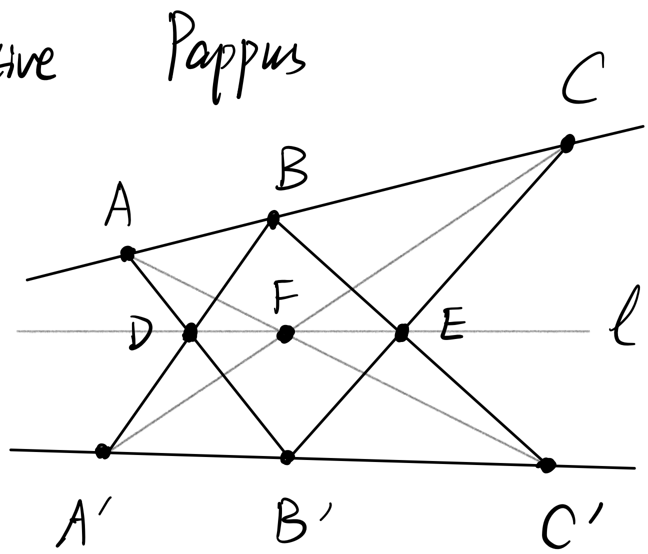
Any plane isometry is determined by images of 3 points.

(4-th point determination))

Project a line to the horizon:

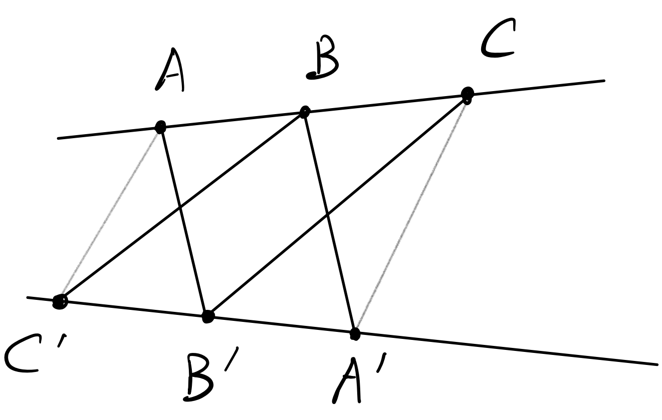


Projective Pappus

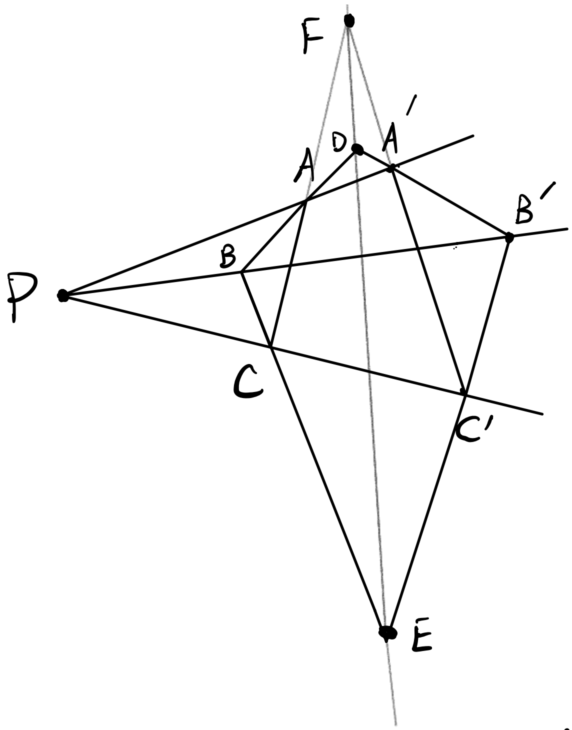


$$\left.\begin{array}{l} AB' \cap A'B = D \\ BC' \cap B'C = E \end{array}\right\} \Rightarrow AC' \cap A'C = F \text{ lines on the line } \overline{DE}.$$

Project l to the horizon



$$\left.\begin{array}{l} AB' \cap BA' \in \text{horizon} \Leftrightarrow AB' /\!/ BA' \\ BC' \cap B'C \in \text{horizon} \Leftrightarrow BC' /\!/ B'C \end{array}\right\} \Rightarrow AC' /\!/ A'C \Leftrightarrow \underset{\text{horizon}}{\underset{\cap}{AC' \cap A'C}}$$



Projective Desargues

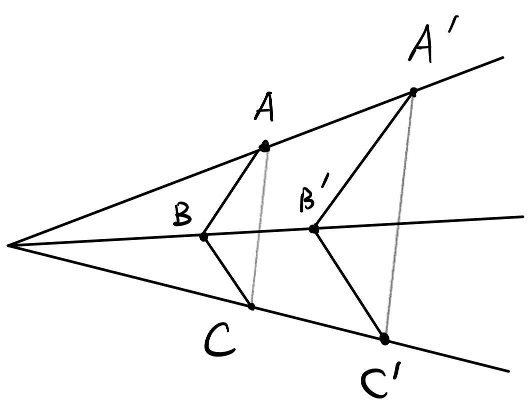
ΔABC and $\Delta A'B'C'$ in perspective from P

line $BA \cap B'A' = D$

$BC \cap B'C' = E$

$\Rightarrow AC \cap A'C' \in$ the line DE

Project the line DE to the horizon:



$AB \cap A'B' \in$ horizon $\Leftrightarrow AB // A'B'$

$BC \cap B'C' \in$ horizon $\Leftrightarrow BC // B'C'$

$\Rightarrow AC \cap A'C' \in$ horizon $\Leftrightarrow AC // A'C'$.