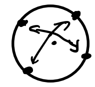
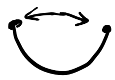
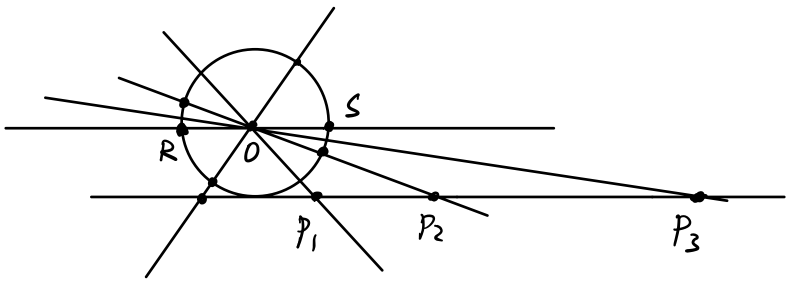
$\mathbb{R}P^1$: Projective line $= \mathbb{R} \cup \{\infty\}$

$= \{\text{lines passing through } 0 \in \mathbb{R}^2\}$

$= S^1 /$ gluing antipodal points 

$=$  $\cong S^1$



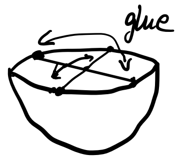
As $P_i \to \infty$, OP_i converges to the horizontal line that cuts the circle at two antipodal points that both represents $\infty \in \mathbb{R}P^1$.

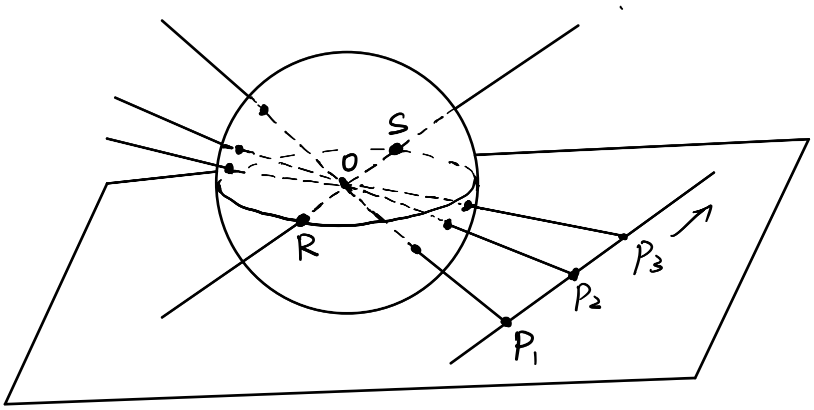
Projective Plane $= \mathbb{R}^2 \cup \mathbb{R}P^1$ ⇐ the "horizon" line = line at infinity

$= \{\text{lines passing through } 0 \in \mathbb{R}^3\}$

$= S^2 /$ gluing antipodal points

glue antipodal points on the equator

$=$ 



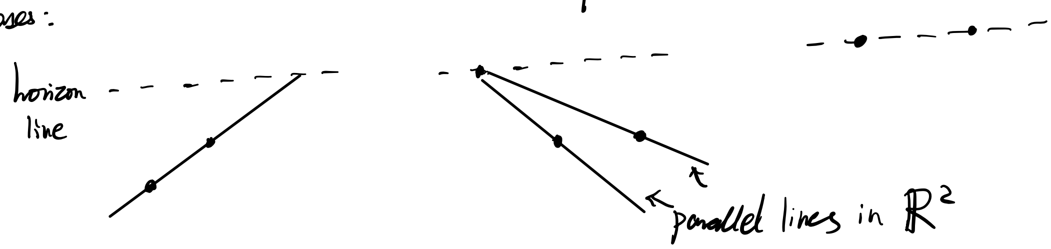
The line $\overline{OP_i}$ converges to the line $\overline{ROS}$ that is parallel to $\overline{P_1P_2}$

In $\mathbb{RP}^2$: "point" $\Longleftrightarrow$ line passing through $O \in \mathbb{R}^3$

"line" $\Longleftrightarrow$ plane passing through $O \in \mathbb{R}^3$

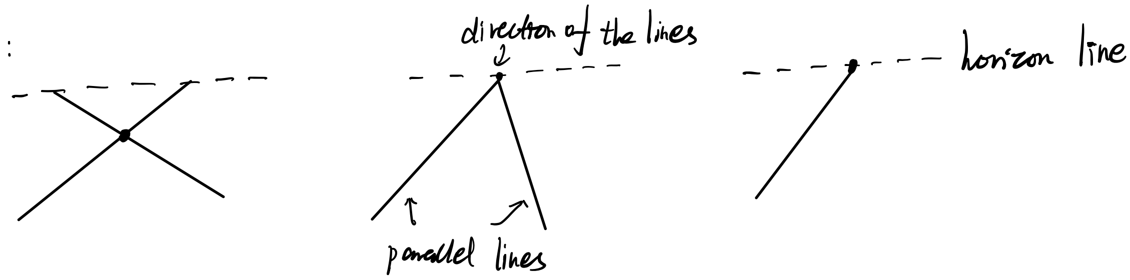
"Two points lie on a unique line" $\Longleftrightarrow$ two lines passing through $O \in \mathbb{R}^3$ spans a unique plane

3 cases:



"Two lines intersect at a unique point" $\Longleftrightarrow$ two planes passing through $O \in \mathbb{R}^2$ intersect at a unique line.

3 cases:



homogeneous coordinates $\Longleftrightarrow$ the line $\mathbb{R}\cdot(a,b,c)$: $\begin{cases} x = a\cdot t \\ y = b\cdot t \\ z = c\cdot t \end{cases}$ $t\in\mathbb{R}$

of a point P

$[a, b, c]$

dual homogeneous coordinates $\Longleftrightarrow$ the plane $Ax+By+Cz=0$.

of a line L: $[A, B, C]$

P lies on L $\Longleftrightarrow$ $A\cdot a + B\cdot b + C\cdot c = 0$.

i.e. $P \in L$

Linear algebra explanation:

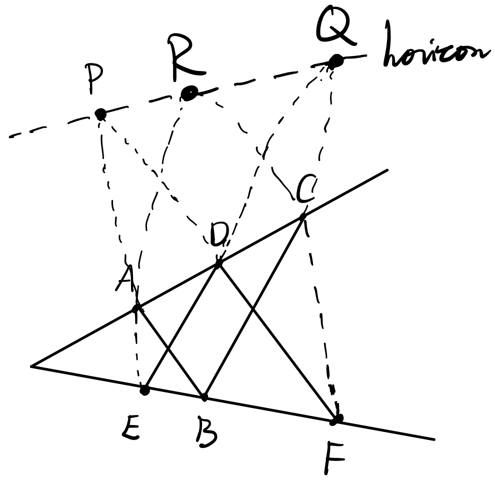
2 points $[a_1, b_1, c_1], [a_2, b_2, c_2]$ lie on a unique line $\Longleftrightarrow$ The solutions (A,B,C) of $\mathbb{R}^3$ $\begin{cases} A\cdot a_1 + B\cdot b_1 + C\cdot c_1 = 0 \\ A\cdot a_2 + B\cdot b_2 + C\cdot c_2 = 0 \end{cases}$ form a 1-dim vector space

$[a_1, b_1, c_1] \neq [a_2, b_2, c_2] \Longleftrightarrow \operatorname{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$.

Similarly:

2 lines $[A_1, B_1, C_1], [A_2, B_2, C_2]$ intersect at a unique point $\Longleftrightarrow$ The solutions (x, y, z) of $\begin{cases} A_1 x + B_1 y + C_1 z = 0 \\ A_2 x + B_2 y + C_2 z = 0 \end{cases}$ form a 1-dim subspace of (a line)

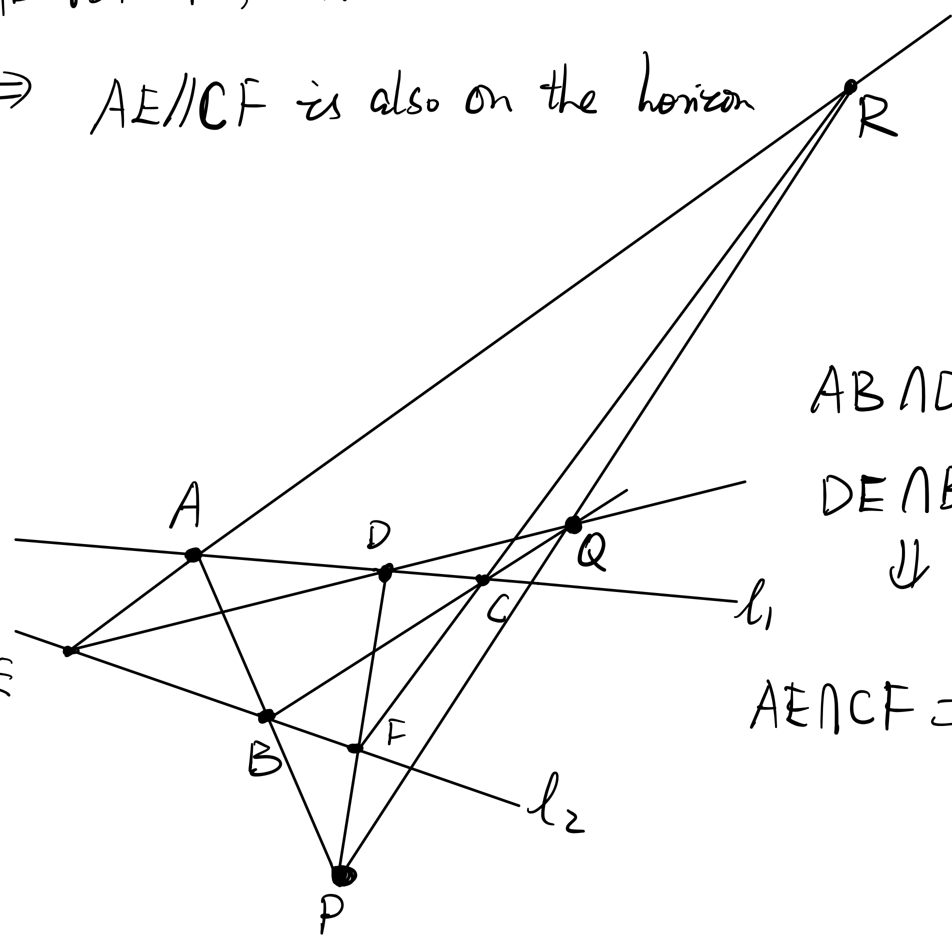
Pappus Thm:



$AB /\!/ DF, \ DE /\!/ BC \Rightarrow AE /\!/ DF$

$\Leftrightarrow AB \cap DF = P, \ DE \cap BC = Q$ on the horizon

$\Rightarrow AE /\!/ CF$ is also on the horizon



$AB \cap DF = P$

$DE \cap BC = Q$

$\Downarrow$

$AE \cap CF = R \in \overline{PQ}$