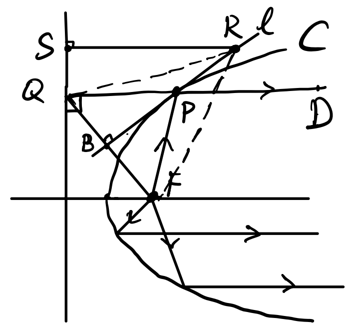
Tangents to conic curves:

parabola:



bisection PB of $\angle FPQ$ = tangent of the parabola C at P.

Need to show the line $l = BP$ lies "outside" of C.

It suffices to show $\forall R \in \ell$, $R \neq P$, $|RF| > |RS|$.

By SAS, $\Delta_{QPB} \cong \Delta_{FPB}$. So $|BQ| = |BF|$.

By SAS, $\Delta_{FBR} \cong \Delta_{QBR} \Rightarrow |QR| = |FR|$

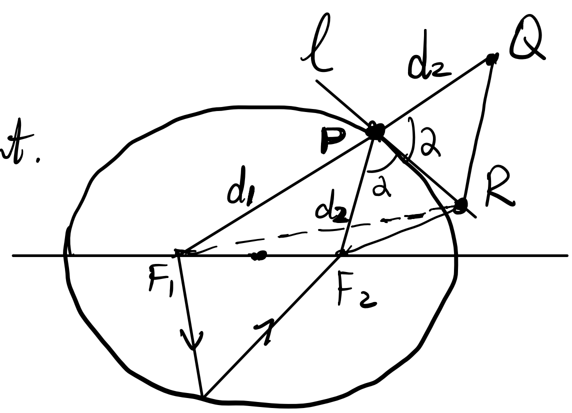
In the right triangle Δ_{QSR}, $|QR| > |SR|$.

So $|RF| = |QR| > |RS|$.

This explains why the lights shooted from the focus F are reflected to be all horizontal.

Ellipse:

$d_1 + d_2 = \text{constant} = 2a$.



bisection l of $\angle F_2PQ$ = tangent line at P:

$\forall R \in l$, $R \neq P$, SAS $\Rightarrow \triangle F_2PR \cong \triangle QPR$

$\Rightarrow |F_2R| = |RQ|$.

In the triangle $\triangle F_1RQ$, $|F_1R| + |RQ| > |F_1Q|$

$\Rightarrow |F_1R| + |F_2R| = |F_1R| + |QR| > |F_1Q| = d_1 + d_2 = 2a$.

$\Rightarrow$ The line l lies on the outside of ellipse

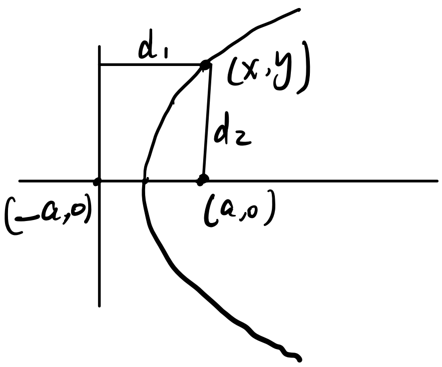
$\Rightarrow$ l is tangent to ellipse at P.

$\Rightarrow$ Fact: Lights from one focus will be reflected to pass through the other focus.

Exercise: Do the same discussion for the hyperbola.

- Equations for conic curves

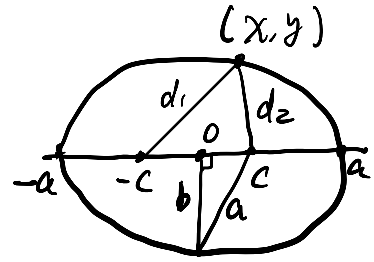
parabola:



$$d_1 = d_2 \iff |x+a| = \sqrt{(x-a)^2+y^2}$$

$$\iff \underset{\substack{\parallel \\ x^2+2ax+a^2}}{(x+a)^2} = \underset{\substack{\parallel \\ x^2-2ax+a^2+y^2}}{(x-a)^2+y^2}$$

$$\iff \boxed{4a\cdot x = y^2}$$



- Ellipse: $d_1+d_2=2a$

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\Updownarrow$$

$$(x+c)^2+y^2 = \left(2a-\sqrt{(x-c)^2+y^2}\right)^2 = 4a^2-4a\cdot\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

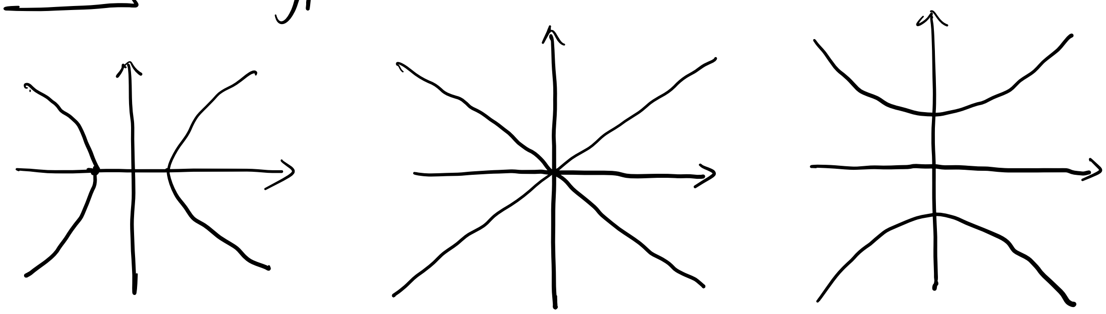
$$\Updownarrow$$

$$4a^2-4c\cdot x = 4a\cdot\sqrt{(x-c)^2+y^2} \iff \underset{\parallel}{(a^2-cx)^2} = a^2\cdot\left((x-c)^2+y^2\right)$$

$$\underset{\substack{\parallel \\ b^2}}{(a^2-c^2)}x^2+a^2y^2 = a^4-a^2c^2 \iff a^4-2a^2c\cdot x+c^2\cdot x^2 = a^2(x^2-2cx+c^2+y^2)$$

$$b^2x^2+a^2y^2 = a^2\cdot b^2 \quad (\text{with } b^2=a^2-c^2) \iff \boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

__Exercise:__ hyperbola:



$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = \left(\frac{x}{a}+\frac{y}{b}\right)\cdot\left(\frac{x}{a}-\frac{y}{b}\right)$$

Classification of conic curves by equations:

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0. \quad (*)$$

step 1. eliminate the xy-term:

$$Ax^2+2Bxy+Cy^2=(x\ y)\begin{pmatrix}A & B\\ B & C\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=X^T\cdot M\cdot X$$

where $X=\begin{pmatrix}x\\ y\end{pmatrix}$, $M=\begin{pmatrix}A & B\\ B & C\end{pmatrix}$

Use rotation to change coordinate:

$$\begin{pmatrix}x\\ y\end{pmatrix}=S\cdot\begin{pmatrix}u\\ v\end{pmatrix} \text{ with } S=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \Rightarrow \begin{matrix}S^TS=I_2\\ \det S=1.\end{matrix}$$

$$X = S\cdot U$$

$\Rightarrow \quad X^T M X = U^T (S^T M S) U$

Find S s.t. $\quad S^T M S = S^{-1} M S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

linear algebra Fact: Any symmetric matrix can be diagonalized by orthogonal matrices.

After change of coordinates $\Leftrightarrow$ rotation of axis: $\quad x$-axis $\rightsquigarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

y-axis $\rightsquigarrow \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

The equation $(*)$ $\rightsquigarrow$

$$\lambda_1 u^2 + \lambda_2 v^2 + 2pu + 2qv + r = 0.$$

Here $\quad 2Dx + 2Ey = 2(D\ E)\begin{pmatrix} x \\ y \end{pmatrix} = 2(D\ E)\, S \begin{pmatrix} u \\ v \end{pmatrix} = 2(p\ q)\begin{pmatrix} u \\ v \end{pmatrix}$

step 2 $\quad$ Complete the squares (when $\lambda_1\lambda_2 \neq 0$)

$$\lambda_1 \left(u + \frac{p}{\lambda_1}\right)^2 + \lambda_2 \left(v + \frac{q}{\lambda_2}\right)^2 = -r + \frac{p^2}{\lambda_1} + \frac{q^2}{\lambda_2} = t$$

step 3 $\quad$ calculate the center: $\quad (u_0, v_0) = \left(-\frac{p}{\lambda_1}, -\frac{q}{\lambda_2}\right)$

$\Rightarrow \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = S \cdot \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ in the original (x, y) coordinate

Write down the axis: $\quad$ direction $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

classification:

Case 1: $\underset{\underset{\det M}{\parallel}}{\lambda_1 \cdot \lambda_2} > 0$. can assume $\lambda_1 > 0, \lambda_2 > 0$

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = S \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

Set $z_1 = u + \frac{p}{\lambda_1}$, $z_2 = v + \frac{q}{\lambda_2}$ $\Rightarrow \lambda_1 z_1^2 + \lambda_2 z_2^2 = t$

1a: $t > 0$: ellipse with center $\left(-\frac{p}{\lambda_1}, -\frac{q}{\lambda_2}\right) = (u_0, v_0)$

axis directions are given by the columns of S: $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

1b: $t = 0$: point $\left(-\frac{p}{\lambda_1}, -\frac{q}{\lambda_2}\right) = (u_0, v_0) \rightsquigarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = S \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

1c: $t < 0$: empty set (imginary ellipse)

Case 2: $\lambda_1 \lambda_2 < 0$.

2a: $t \neq 0$: hyperbola. (center and axis similar to above)

2b: $t = 0$: 2 lines

Case 3: $\lambda_1 \lambda_2 = 0$. Assume $\lambda_1 = 0$, $\lambda_2 \neq 0$.

$$(*) \Rightarrow \lambda_2 v^2 + 2pu + 2qv + r = 0$$

$$\Updownarrow$$

$$\lambda_2 \cdot \left(v + \frac{q}{\lambda_2}\right)^2 = \underbrace{-2pu - r + \frac{q^2}{\lambda_2}}_{\displaystyle = -2p\left(u - \frac{1}{2p}\left(\frac{q^2}{\lambda_2} - r\right)\right)}$$

$$z_2^2 = -\frac{2p}{\lambda_2} z_1$$

If $p \neq 0$, then this is a parabola

with vertex: $(u_0\ v_0) = \left(\frac{1}{2p}\left(\frac{q^2}{\lambda_2} - r\right), -\frac{q}{\lambda_2}\right)$

$\rightsquigarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = S\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

and axis direction: $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

Focus: $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \frac{2p}{4\lambda_2} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

If $p = 0$: $\lambda_2 v^2 + 2qv + r = 0 \Rightarrow$ lines or empty set.

Can also find center first when $\lambda_1\lambda_2 \neq 0$: Find translation

$\begin{cases} x' = x - x_0, \\ y' = y - y_0 \end{cases}$ that elliminates the linear term first:

$\rightsquigarrow A\cdot(x'+x_0)^2 + 2B(x'+x_0)\cdot(y'+y_0) + C\cdot(y'+y_0)^2 + 2D\cdot(x'+x_0) + 2E(y'+y_0) + F = 0$

$=$ Quadratic terms $+ x'(2A\cdot x_0 + 2By_0 + 2D) + y'(2Bx_0 + 2C\cdot y_0 + 2E) + F'$

$\rightsquigarrow \begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -\begin{pmatrix} D \\ E \end{pmatrix} \Rightarrow$ Center given by $-M^{-1}\begin{pmatrix} D \\ E \end{pmatrix}$

if $\underset{\lambda_1\lambda_2}{\underset{\parallel}{\det M}} \neq 0$.

Ex 1: $36x^2-24xy+29y^2+120x-290y+545=0$

(step 1) diagonalize $M=\begin{pmatrix}36 & -12\\ -12 & 29\end{pmatrix}$ by orthogonal matrix S with $\det S=1$.

$|\lambda I-M|=\begin{vmatrix}\lambda-36 & 12\\ 12 & \lambda-29\end{vmatrix}=\lambda^2-65\lambda+\underbrace{36\cdot 29-12\cdot 12}_{1044-144}$

$\begin{array}{r}36\\ 29\\ \hline 324\\ 72\\ \hline 1044\end{array}$

$(\lambda-20)\cdot(\lambda-45)=\lambda^2-65\lambda+900$

$\Rightarrow \lambda=20,\ 45.$

$\lambda=20$: $\lambda I-M=\begin{pmatrix}-16 & 12\\ 12 & -9\end{pmatrix}\to\begin{pmatrix}4 & -3\\ 4 & -3\end{pmatrix}$

$\Rightarrow$ eigenvector $\tilde{f}_1=\begin{pmatrix}3\\4\end{pmatrix}$ $\rightsquigarrow$ normalize $f_1=\frac{1}{\sqrt{3^2+4^2}}\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$

$\lambda=45$: eigenvector $f_2\perp f_1$, $\det(f_1 f_2)=1$ $\Rightarrow$ $f_2=\begin{pmatrix}-\frac{4}{5}\\ \frac{3}{5}\end{pmatrix}$

$\Rightarrow X=M\cdot U$ with $M=(f_1\ f_2)=\begin{pmatrix}\frac{3}{5} & -\frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{pmatrix}$

$36x^2-24xy+29y^2\rightsquigarrow 20u^2+45v^2$

$120x-290y=(120\ \ -290)\begin{pmatrix}x\\y\end{pmatrix}=(120\ \ -290)\cdot\begin{pmatrix}\frac{3}{5} & -\frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$

$=\frac{1}{5}\cdot(\underbrace{120\cdot 3-290\cdot 4}_{360-1160=-800},\ \underbrace{-120\cdot 4-290\cdot 3}_{-480-870=-1350})=(-160,\ -270)$

$$\Rightarrow \quad 20u^2+45v^2-160u-270v+545=0$$

$$\Updownarrow$$

(step 2: complete the squares)

$$4u^2+9v^2-32u-54v+109=0 \qquad -109+145$$

$$\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad\qquad \|$$

$$4(u^2-8u+16)+9(v^2-6v+9)=\underbrace{-109+64+81}_{=36}$$

$$\|$$

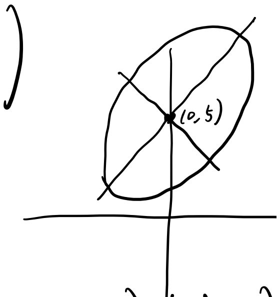
$$4(u-4)^2+9(v-3)^2=36$$

Set $\begin{cases} z_1=u-4 \\ z_2=v-3 \end{cases}$: $\quad 4z_1^2+9z_2^2=36 \iff \dfrac{z_1^2}{9}+\dfrac{z_2^2}{4}=1.$

ellipse with center $\underbrace{(4,3)}_{=(u_0,v_0)}$. axis direction $\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}, \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$.

(Step 3: calculate the center. Write down the axis)

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = M\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$



<u>Center first</u>: $-M^{-1}\begin{pmatrix} D \\ E \end{pmatrix} = -\begin{pmatrix} 36 & -12 \\ -12 & 29 \end{pmatrix}^{-1}\begin{pmatrix} -\frac{120}{2} \\ \frac{290}{2} \end{pmatrix} = -\frac{1}{2\cdot 900}\begin{pmatrix} 29 & 12 \\ 12 & 36 \end{pmatrix}\cdot\begin{pmatrix} -120 \\ 290 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 5 \end{pmatrix} = -\frac{1}{2\cdot 10}\begin{pmatrix} 0 \\ \underbrace{-16+4\cdot 29}_{=-16+116} \end{pmatrix} = -\frac{1}{900}\begin{pmatrix} 0 \\ 12\cdot(-120)+36\cdot 290 \end{pmatrix}$$

Ex 2: $16x^2-24xy+9y^2-130x-90y+50=0.$

Step 1. $M=\begin{pmatrix}16 & -12\\ -12 & 9\end{pmatrix}$ $\quad |\lambda I-M|=\begin{vmatrix}\lambda-16 & 12\\ 12 & \lambda-9\end{vmatrix}=\lambda^2-25\lambda=\underbrace{\lambda(\lambda-25)}_{=0}$

$\Rightarrow \lambda_1=0,\ \lambda_2=25$

$\lambda_1=0:\ \lambda I-M=-M=\begin{pmatrix}-16 & 12\\ 12 & -9\end{pmatrix}\to\begin{pmatrix}4 & -3\\ 0 & 0\end{pmatrix}\Rightarrow \tilde{f}_1=\begin{pmatrix}3\\4\end{pmatrix}$

$\Rightarrow f_1=\frac{\tilde{f}_1}{|\tilde{f}_1|}=\frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix}$

$\Rightarrow S=\frac{1}{5}\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}$

$\lambda_2=25:\ f_2=\frac{1}{5}\begin{pmatrix}-4\\3\end{pmatrix}.$

$16x^2-24xy+9y^2\longrightarrow 25\cdot v^2$

$-130x-90y=\begin{pmatrix}-130 & -90\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

$\longrightarrow \begin{pmatrix}-130 & -90\end{pmatrix}\frac{1}{5}\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$

Step 2

$= -2\cdot(13\cdot 3+9\cdot 4 \quad 13\cdot(-4)+9\cdot 3)$

$= -2\cdot(39+36,\ -52+27)=-2\cdot(75,-25)=(-150,50)$

$\leadsto\ 25v^2-150u+50v+50=0$

$\Updownarrow$

$v^2-6u+2v+2=0$

$$\Leftrightarrow \quad \underset{\underset{(v+1)^2}{\shortparallel}}{v^2+2v+1} = 6u-2+1 = 6\cdot\left(u-\frac{1}{6}\right)$$

Step 3

$\Rightarrow$ vertex $(u_0\ v_0) = \left(\frac{1}{6}, -1\right)$

$$\Rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = S \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{5}\cdot\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{6} \\ -1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} \frac{1}{2}+4 \\ \frac{3}{2}-3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} \frac{9}{2} \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{9}{10} \\ -\frac{3}{10} \end{pmatrix}$$

axis direction: $\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$