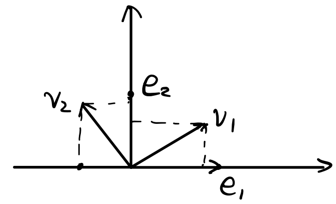
- rotations around $0 \in \mathbb{R}^2$

$$e_1 \mapsto v_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$e_2 \mapsto v_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

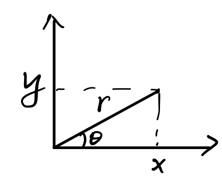
$$x_1 e_1 + x_2 e_2 \mapsto x_1 v_1 + x_2 v_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_\theta x = r_\theta(x)$$

A is orthogonal : $A^T \cdot A = I_2$. $\det A = \cos^2\theta + \sin^2\theta = 1$.

$$\begin{aligned} r_{\theta_1+\theta_2}(x) &= r_{\theta_1} \circ r_{\theta_2}(x) \\ \| \quad & \\ A_{\theta_1+\theta_2} \cdot x &= A_{\theta_1} A_{\theta_2} x \end{aligned} \iff \begin{cases} \cos(\theta_1+\theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \\ \sin(\theta_1+\theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 \end{cases}$$

- complex numbers $z = x + iy = r \cdot e^{i\theta}$, $i^2 = -1$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{aligned} e^{i\alpha}(x+iy) &= (\cos\alpha + i\sin\alpha) \cdot (x+iy) \\ \| \qquad\quad &= (\cos\alpha \cdot x - \sin\alpha \cdot y) + i \cdot (\sin\alpha \cdot x + \cos\alpha \cdot y) \\ e^{i\alpha} \cdot re^{i\theta} &= r \cdot e^{i(\alpha+\theta)} \end{aligned}$$

multiplication by $e^{i\alpha}$

$\Updownarrow$

rotation by the angle α.

• <u>Thm</u>: Let f be an isometry of $\mathbb{R}^2$. Then there exists an orthogonal (2x2) matrix A and a vector $b \in \mathbb{R}^2$ s.t.

$$f(x) = A \cdot x + b \quad \forall x \in \mathbb{R}^2.$$

<u>Proof</u>: Assume $f(0) = b$. Define $g(x) = f(x) - b$.

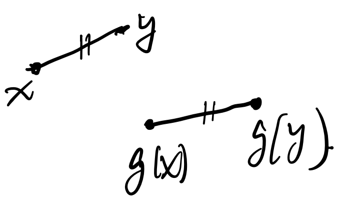
It suffices to show that $g(x) = Ax$ for some orthogonal matrix A.

In particular, we need to show that $g(x)$ is linear:

$$g(x+y) = g(x) + g(y), \quad g(ax) = a\, g(x) \quad \forall x, y \in \mathbb{R}^2, a \in \mathbb{R}.$$

. g is isometry $\Leftrightarrow |g(x) - g(y)| = |x - y|, \ \forall x, y \in \mathbb{R}^2$

$$\begin{array}{ccc} |g(x) - g(y)| & = & |x-y| \\ \| & & \| \\ |\overline{g(x)\, g(y)}| & & |\overline{xy}| \end{array}$$



In particular, $|g(x)| = |x|, \ \forall x \in \mathbb{R}^2$.

Because the inner product $x \cdot y = (|x|^2 + |y|^2 - |x-y|^2)/2$,

we conclude that $g(x) \cdot g(y) = x \cdot y$.

$$\frac{|g(x)|^2 + |g(y)|^2 - |g(x) - g(y)|^2}{2} = \frac{|x|^2 + |y|^2 - |x-y|^2}{2}$$

Set $v_1 = g(e_1) = g\begin{pmatrix}1\\0\end{pmatrix}$, $v_2 = g(e_2) = g\begin{pmatrix}0\\1\end{pmatrix}$.

Then $\left.\begin{array}{l} v_1 \cdot v_2 = (1,0)\cdot\begin{pmatrix}1\\0\end{pmatrix} = 0 \Rightarrow v_1 \perp v_2 \\ v_i \cdot v_i = 1^2 + 0^2 = 1 \Rightarrow |v_i| = 1 \end{array}\right\} \Rightarrow \{v_1, v_2\}$ is an orthonormal basis.

$$\begin{aligned}
\Rightarrow g(x) &= (g(x)\cdot v_1)v_1 + (g(x)\cdot v_2)v_2 \\
&= (g(x)\cdot g(e_1))v_1 + (g(x)\cdot g(e_2))v_2 \\
&= (x\cdot e_1)g(e_1) + (x\cdot e_2)g(e_2) \qquad \forall x = (x\cdot e_1)e_1 + (x\cdot e_2)e_2
\end{aligned}$$

$\Rightarrow g(x)$ is linear. Moreover if $v_i = a_i e_1 + b_i e_i$, then

$= (e_1\ e_2)\begin{pmatrix}a_i\\b_i\end{pmatrix}$

$$g(x) = \begin{pmatrix} g(x)\cdot e_1 \\ g(x)\cdot e_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} x\cdot e_1 \\ x\cdot e_2 \end{pmatrix} = A\cdot x$$

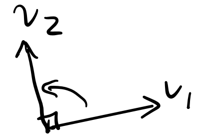
$\{v_1, v_2\}$ is an orthonormal basis $\Longleftrightarrow A = (v_1\ v_2)$ is an orthogonal matrix

i.e. $A^T A = I_2$ ■

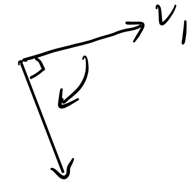
$A^TA = I_2 \Rightarrow \{v_1, v_2\}$ is an orthonormal basis

$A = (v_1, v_2)$

$\det A = 1 \Rightarrow \{v_1, v_2\}$ is positively oriented (right-handed)



$\det A = -1 \Rightarrow \{v_1, v_2\}$ is negatively oriented



- Classification of plane isometry by linear algebra.

$$f = Ax + b, \quad A^TA = I_2, \quad b \in \mathbb{R}^2.$$

1. f is a translation $\Longleftrightarrow A = I_2$
2. f is a rotation $\Longleftrightarrow \det(A) = 1$ and $A \neq I_2$

center of rotation x_0 satisfies:

$$A x_0 + b = x_0 \Rightarrow x_0 = (I - A)^{-1} b.$$

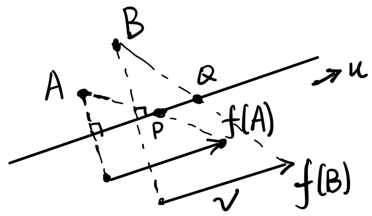
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \text{characteristic polynomial } |\lambda I - A| = \lambda^2 - 2\cos\theta + 1$$

$$\Rightarrow \text{eigenvalues: } \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 1}}{2} = \cos\theta \pm i\sin\theta.$$

$\Rightarrow I - A$ is invertible if $\theta \neq 0$. i.e. if $A \neq I_2$.

3. f is a glide reflection if $\det(A) = -1$.

$$\left(\begin{array}{c} A^T A = I_2 \\ \Downarrow \\ (\det A)^2 = 1 \\ \Downarrow \\ \det A = \pm 1 \end{array}\right)$$



$$A = (v_1 \ \ v_2) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

characteristic polynomial: $P(\lambda) = \begin{vmatrix} \lambda - \cos\theta & -\sin\theta \\ -\sin\theta & \lambda + \cos\theta \end{vmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$

$\Rightarrow$ eigenvalues $\lambda = 1, -1$.

$\lambda = 1 \rightarrow$ eigenvector u $\quad Au = u$

$\lambda = -1 \rightarrow$ eigenvector u' $\quad Au' = -u'$

$$u \cdot u' = u^T u' = u^T (A^T \cdot A) u' = (Au)^T \cdot Au' = \underset{\underset{-u \cdot u'}{\shortparallel}}{u^T \cdot (-u')}$$

$\Rightarrow u \cdot u' = 0 \Leftrightarrow u \perp u'$

$x = au + bu'$, $\quad (A+I)x = Ax + x = au - bu' + au + bu' = 2au$

$\quad (A-I)x = Ax - x = au - bu' - (au + bu') = -2bu'$

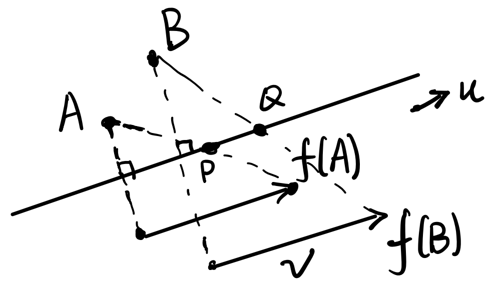
$\Rightarrow \quad \mathrm{Im}(A+I) = \mathrm{Span}\{u\} = \ker(A-I) = \mathrm{Span}\{u\}$

$\mathrm{Im}(A-I) = \mathrm{span}\{u'\} = \ker(A+I) = \mathrm{Span}\{u'\}$

$$\underset{\underset{\mathrm{Im}(A+I)}{\shortparallel}}{\ker(A-I)} \perp \underset{\underset{\mathrm{Im}(A-I)}{\shortparallel}}{\ker(A+I)} \qquad \mathbb{R}^2 = \mathbb{R}u \oplus \mathbb{R}u'$$

Axis of reflection:

$$\overrightarrow{PQ} = \frac{1}{2}\cdot(f(y)+y) - \frac{1}{2}(f(x)+x)$$
$$= \frac{1}{2}(f(y)-f(x)+(y-x))$$
$$= \frac{1}{2}\cdot(Ay+b-(Ax+b)+(y-x))$$
$$= \frac{1}{2}(A+I)(y-x).$$



direction of axis : $u \in \mathrm{Im}(A+I) = \mathrm{Ker}(A-I)$

A point on the axis: $\frac{1}{2}(0 + A(0)) = \frac{b}{2}$

translation vector :

$$v = \mathrm{Proj}_{\frac{u}{|u|}}(f(x)-x) = \left((f(x)-x)\cdot\frac{u}{|u|}\right)\frac{u}{|u|} = \frac{(f(x)-x)\cdot u}{|u|^2}u$$

$$= \frac{(Ax+b-x)\cdot u}{|u|^2}u = \frac{(A-I)x\cdot u}{|u|^2} + \frac{b\cdot u}{|u|^2}u = \frac{b\cdot u}{|u|^2}u$$

$\left((A-I)x\cdot u = 0 \quad \text{because} \quad \mathrm{Im}(A-I) \perp \mathrm{Ker}(A-I) \right)$

Ex: $f(0,0)=(1,1)$, $f(1,0)=(1.8,1.6)$, $f(0,1)=(0.4,1.8)$.

classify f: use column vectors. set $b=f(0)=\begin{pmatrix}1\\1\end{pmatrix}$.

Set $g(x)=f(x)-f(0)=f(x)-b$. Then

$$g(\underline{0})=\underline{0},\quad g(e_1)=\begin{pmatrix}0.8\\0.6\end{pmatrix},\quad g(e_2)=\begin{pmatrix}-0.6\\0.8\end{pmatrix}$$

$$\Rightarrow g(x)=\begin{pmatrix}0.8 & -0.6\\0.6 & 0.8\end{pmatrix}x=A\cdot x$$

$\det A=(0.8)^2+(0.6)^2=1 \Rightarrow f$ is a rotation.

Angle of rotation: $\theta=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)=\tan^{-1}\left(\frac{0.6}{0.8}\right)=\tan^{-1}\left(\frac{3}{4}\right)$.

center of rotation: $x_0=(I-A)^{-1}b$

$$I-A=\begin{pmatrix}0.2 & 0.6\\-0.6 & 0.2\end{pmatrix}\Rightarrow (I-A)^{-1}=\frac{1}{0.2^2+0.6^2}\begin{pmatrix}0.2 & -0.6\\0.6 & 0.2\end{pmatrix}$$

$$=\frac{1}{0.4}\begin{pmatrix}0.2 & -0.6\\0.6 & 0.2\end{pmatrix}$$

$$=\begin{pmatrix}0.5 & -1.5\\1.5 & 0.5\end{pmatrix}$$

$$\Rightarrow x_0=\begin{pmatrix}0.5 & -1.5\\1.5 & 0.5\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\2\end{pmatrix}.$$

Ex: $f(0,0)=(1,1)$, $f(1,0)=(0.4,1.8)$, $f(0,1)=(1.8,1.6)$

classify f:

Let $g=f-f(0)$. Then $g(e_1)=\begin{pmatrix}-0.6\\0.8\end{pmatrix}$, $g(e_2)=\begin{pmatrix}0.8\\0.6\end{pmatrix}$

$g(x)=f(x)-f(0)=A\cdot x$ where $A=\begin{pmatrix}-0.6 & 0.8\\0.8 & 0.6\end{pmatrix}$

- $\det A=-0.6^2-0.8^2=-1<0 \Rightarrow f$ is a glide reflection.

- Find axis: $|\lambda I-A|=\begin{vmatrix}\lambda+0.6 & -0.8\\-0.8 & \lambda-0.6\end{vmatrix}=\lambda^2-1=0 \Rightarrow \lambda=\pm1.$

$\lambda=1$: $I-A=\begin{pmatrix}1.6 & -0.8\\-0.8 & 0.4\end{pmatrix}\to\begin{pmatrix}2 & -1\\0 & 0\end{pmatrix}\Rightarrow u=\begin{pmatrix}1\\2\end{pmatrix}.$

Axis passes through $\frac{b}{2}=\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$

$\Rightarrow$ Axis: $\begin{cases}x=\frac{1}{2}+t\\y=\frac{1}{2}+2t\end{cases} \Leftrightarrow 2(x-\frac{1}{2})=y-\frac{1}{2}$

$\Updownarrow$

$y=2x-\frac{1}{2}.$

- translation vector:

$$v=\frac{b\cdot u}{|u|^2}u=\frac{1\cdot1+1\cdot2}{1^2+2^2}\cdot\begin{pmatrix}1\\2\end{pmatrix}=\frac{3}{5}\begin{pmatrix}1\\2\end{pmatrix}.$$