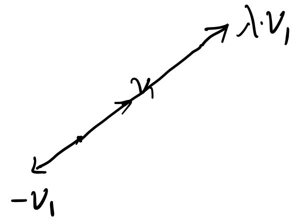
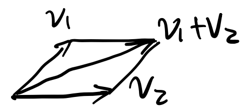
Vector calculus

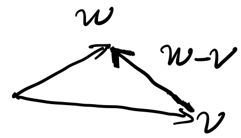
$\mathbb{R}^2$: vector space



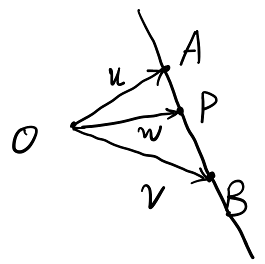
scalar multiplication



vector addition



Application :



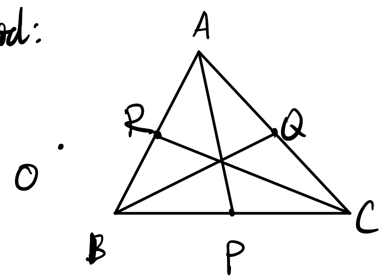
$P \in$ line $AB \iff \vec{AP} \parallel \vec{AB}$

$\iff w-u = \lambda(v-u)$

$\iff w = (1-\lambda)u + \lambda v$

$\iff w = au+bv$ with $a+b=1$.

Centroid:



$\vec{OA} = u, \quad \vec{OB} = v, \quad \vec{OC} = w$

$\Rightarrow \vec{OP} = \frac{1}{2}(v+w), \quad \vec{OQ} = \frac{1}{2}(u+w)$

$\vec{OR} = \frac{1}{2}(u+v)$

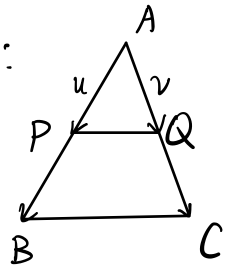
Observe: $\frac{1}{3}(u+v+w) = \frac{1}{3}u + \frac{2}{3}\cdot\frac{1}{2}(v+w)$ $\qquad \frac{1}{3}+\frac{2}{3}=1$

$= \frac{1}{3}v + \frac{2}{3}\cdot\frac{1}{2}(u+w)$

$= \frac{1}{3}w + \frac{2}{3}\cdot\frac{1}{2}(u+v)$

$\Rightarrow$ The end point of $\frac{1}{3}(u+v+w)$ lies on AP, BQ and CR.

Vector Thales Thm:



$\vec{AP} = u,\ \vec{AB} = a \cdot u$

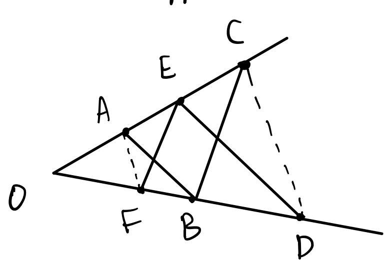
$\vec{AQ} = v,\ \vec{AC} = b v$

If $\vec{PQ} \parallel BC$, then $a = b$

Pf: $\vec{PQ} = v - u \parallel \vec{BC} = bv - au \Rightarrow bv - au = \lambda(v - u)$

$\Rightarrow (\lambda - a) u = (\lambda - b) v \overset{}{\underset{u \nparallel v}{\Longrightarrow}} \lambda - a = 0 = \lambda - b \Rightarrow a = b.$ $\blacksquare$

Vector Pappus Thm:



$\vec{OA} = u,\ \vec{OE} = a \cdot u,\ \vec{OC} = b u$

$\vec{OF} = v,\ \vec{OB} = c \cdot v,\ \vec{OD} = d \cdot v$

If $AB \parallel DE$ and $EF \parallel BC$, then $AF \parallel CD$.

Pf: $AB \parallel DE \Longrightarrow \underset{d \cdot v}{\underset{\parallel}{OD}} = \underset{a \cdot c \cdot v}{\underset{\parallel}{a \cdot OB}} \Rightarrow d = ac$

vector Thales

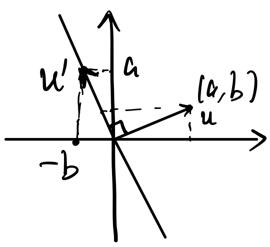
$EF \parallel BC \Longrightarrow \underset{b u}{\underset{\parallel}{\vec{OC}}} = \underset{c \cdot a u}{\underset{\parallel}{c \cdot \vec{OE}}} \Rightarrow b = ac$

$\left.\begin{array}{l} d = ac \\ b = ac \end{array}\right\} \Rightarrow b = d$

$\Rightarrow \vec{CD} = \vec{OD} - \vec{OC} = dv - bu = bv - bu = b(v - u) \parallel v - u = \vec{AF}$ $\blacksquare$

- Inner products

$u=(a,b),\ v=(c,d)$. $\quad u\cdot v = ac+bd$.



$v\perp u \Leftrightarrow v=\lambda\cdot u' = \lambda\cdot(-b,a)$

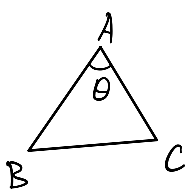
$\Leftrightarrow u\cdot v = (a,b)\cdot\lambda(-b,a) = \lambda(-ab+ba)=0$.

slope of $u=\frac{b}{a}=k$ $\qquad k\cdot k'=-1 \Leftrightarrow u\perp u'$.

slope of $u' = -\frac{a}{b}=k'$

$$\cos\theta = \frac{u\cdot v}{|u||v|} \Leftrightarrow u\cdot v = |u||v|\cos\theta.$$

$$|u-v|^2 = (u-v)\cdot(u-v) = u\cdot u - u\cdot v - v\cdot u + v\cdot v$$
$$= |u|^2+|v|^2-2u\cdot v = |u|^2+|v|^2-2|u||v|\cos\theta$$



$$|BC|^2 = |AB|^2+|AC|^2-2|AB||AC|\cos\theta$$

(law of cosine)

$\theta=\frac{\pi}{2}$: $|BC|^2=|AB|^2+|AC|^2$ (Pythagorean)

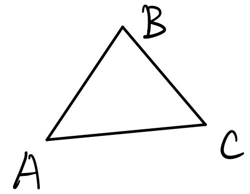
- $|u\cdot v| = |u||v||\cos\theta| \le |u|\cdot|v|$ $\quad\Rightarrow\quad |u-v|^2 = |u|^2+|v|^2-2u\cdot v$

$\qquad\ \ \|\qquad\qquad\qquad\qquad\quad \|\qquad\qquad\qquad\qquad \le |u|^2+|v|^2+2|u||v|$

$|ac+bd| \le \sqrt{a^2+b^2}\sqrt{c^2+d^2}$ $\qquad\qquad = (|u|+|v|)^2$

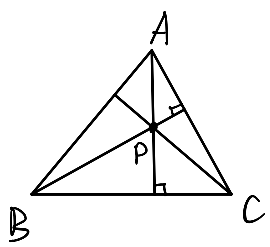
Cauchy-Schwarz

$\Rightarrow \quad |u-v| \leq |u|+|v|$ triangle inequality



$|AC| \leq |AB| + |BC|$

- Altitudes of triangles : 3 altitudes intersect at a common point P



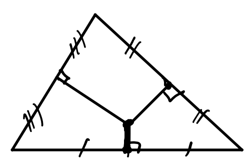
$\vec{AB} = u, \quad \vec{AC} = v, \quad \vec{AP} = t$

$\Rightarrow \vec{BP} = t-u, \quad \vec{CP} = t-v$

$$\left.\begin{aligned} AP \perp BC &\Leftrightarrow \overset{t\cdot v - t\cdot u}{\overset{\parallel}{t\cdot(v-u)}} = 0 \\ BP \perp AC &\Leftrightarrow \underset{t\cdot v - u\cdot v}{\underset{\parallel}{(t-u)\cdot v}} = 0 \end{aligned}\right\} \Rightarrow \underset{(t-v)\cdot u}{\underset{\parallel}{t\cdot u - u\cdot v}} = 0$$

$\Rightarrow \vec{CP} \perp \vec{AB}$ ∎

<u>Problem:</u>



3 equidistant lines intersect at a common point.