Isometries of the Euclid's plane.

Isometry. $f: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $|f(A)f(B)| = |AB|$, $\forall A, B \in \mathbb{R}^2$.

- Composition of isometries is also isometry.

<u>ThmA</u>: Any isometry is a composition of one, two or three reflections.

To prove this theorem, we first prove 2 lemmas.

Pick any three points A, B, C that are not on a line.

<u>Lemma 1</u>: Any point D is uniquely determined by the 3 distances: $\{|DA|, |DB|, |DC|\}$

<u>Pf</u>: Assume a point $E \in \mathbb{R}^2$ satisfies
$$\begin{aligned} |EA| &= |DA| \\ |EB| &= |DB| \\ |EC| &= |DC|. \end{aligned}$$

If $D \neq E$, then A, B, C are on the equidistant line for D and E.

This is a contradiction to the assumption that A, B, C are not on a line.

So $D = E$. $\blacksquare$

<u>Lemma 2</u>: If f and g are isometries s.t. $\begin{cases} f(A) = g(A) \\ f(B) = g(B) \\ f(C) = g(C) \end{cases}$, then $f = g$.

In other words, isometries are uniquely determined by their values (images) of A, B, C.

<u>Pf</u>: A, B, C not on a line $\Rightarrow \triangle ABC$ is a triangle

$\overset{(SSS)}{\Longrightarrow} \triangle f(A)f(B)f(C) = \triangle g(A)g(B)g(C)$ is a triangle

$\Rightarrow A' = f(A) = g(A)$, $B' = f(B) = g(B)$, $C' = f(C) = g(C)$ are not on a line.

$\forall D \in \mathbb{R}^2$, $|f(D)A'| = |f(D)f(A)| = |DA| = |g(D)g(A)| = |g(D)A'|$

Similarly $|f(D)B'| = |g(D)B'|$, $|f(D)C'| = |g(D)C'|$.

By lemma 1, $f(D) = g(D)$. Since this holds for any $D \in \mathbb{R}^2$, $f = g$ ∎

<u>Proof of Thm A</u>: Let r_1 be the reflection that maps $f(A)$ to A

Then $r_1 \circ f$ maps
$$\begin{aligned} A &\to A \\ B &\to r_1 \circ f(B) =: B' \\ C &\to r_1 \circ f(C) = C' \end{aligned}$$
(if $f(A) = A$, then choose $r_1 = id_{\mathbb{R}^2}$)

Because $r_1 \circ f$ is also an isometry, $|AB| = |AB'|$, $|AC| = |AC'|$

Let r_2 be the reflection that maps B' to B. (if $B = B'$, then choose $r_2 = id_{\mathbb{R}^2}$)

Then $r_2 \circ r_1 \circ f$ maps:
$$\begin{aligned} A &\mapsto A \quad (\text{since } |AB| = |AB'|) \\ B &\mapsto B \\ C &\mapsto C'' \end{aligned}$$

Because $r_2 \circ r_1 \circ f$ is an isometry, $|AC| = |AC''|$, $|BC| = |BC''|$.

Let r_3 be the reflection that maps C'' to C (if $C = C''$, then choose $r_3 = id_{\mathbb{R}^2}$)

Then the isometry $r_3 \circ r_2 \circ r_1 \circ f$ maps:
$$\begin{aligned} A &\mapsto A \quad (\text{since } |AC| = |AC''|) \\ B &\mapsto B \quad (\text{since } |BC| = |BC''|). \\ C &\mapsto C \end{aligned}$$

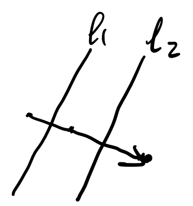
By Lemma 2, $r_3 \circ r_2 \circ r_1 \circ f = id_{\mathbb{R}^2}$.

So $f = r_1 \circ r_2 \circ r_3 \circ r_3 \circ r_2 \circ r_1 \circ f = r_1 \circ r_2 \circ r_3$. ∎

· Composition of two reflections is a translation or rotation.

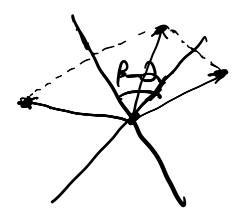
If the axes (mirrors) are:

parallel



translate in perpendicular direction with length $2 \cdot \text{dist}(l_1, l_2)$.

intersect



rotate by the angle $2(\beta - \alpha)$

Observation: If two lines $L \cap M \neq \phi$, then

$r_M \circ r_L = r_{M'} \circ r_{L'}$ for any two lines M', L' that have the same angle as M with L. (r_L is the reflection across L).

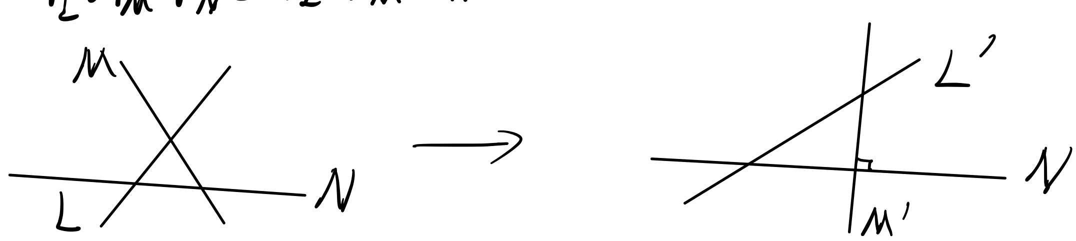
This can be used to prove:

Thm: Any composition of 3 reflections is a glide reflection.

Pf: Let $f = r_L \circ r_M \circ r_N$.

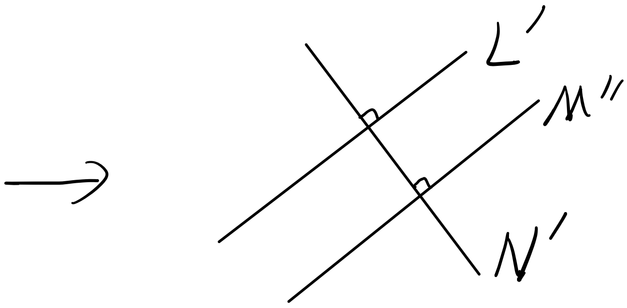
Case 1: $L \cap M \neq \phi$. Modify the axes by (Rubik's cube type):

· $r_L \circ r_M \circ r_N = r_{L'} \circ r_{M'} \circ r_N$ with $M' \perp N$. (rotate axes $L \cup M$)



$r_{L'} \circ r_M \circ r_N = r_{L'} \circ r_{M''} \circ r_{N'}$ with $M'' \parallel L'$ and $M'' \perp N'$

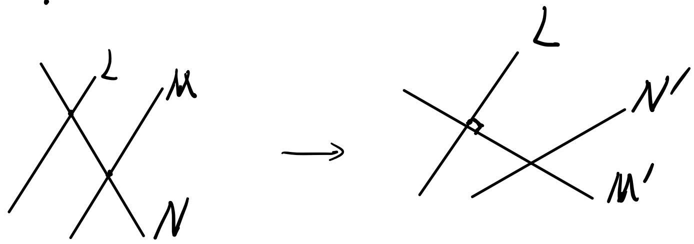
(rotate axes $M' \cup N$)

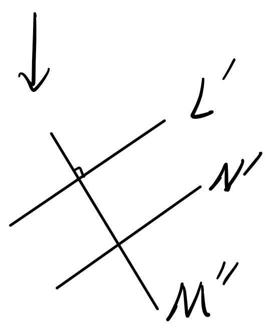


$r_{L'} \circ r_{M''} \circ r_{N'}$ is a glide reflection.

<u>Case 2:</u> $L \parallel M$. If $N \parallel L$, then $r_L \circ r_M \circ r_N$ is a reflection.

Otherwise





$$
\begin{aligned}
r_L \circ r_M \circ r_N &= r_L \circ r_{M'} \circ r_{N'} \\
&= r_{L'} \circ r_{M''} \circ r_{N'} \\
&= \underline{r_{L'} \circ r_{N'} \circ r_{M''}} \text{ glide reflection}
\end{aligned}
$$

∎

Linear algebra characterization:

$f: \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry if and only if there exists a (2×2) orthogonal matrix A: $A^T A = I_2$ and $b = f(0) \in \mathbb{R}^2$

s.t.

$$f(x) = A \cdot x + b \quad \text{for any } x \in \mathbb{R}^2.$$

Q: Determine the type of isometry from A and b.