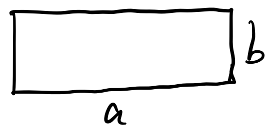
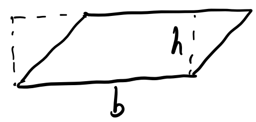
Area: rectangle

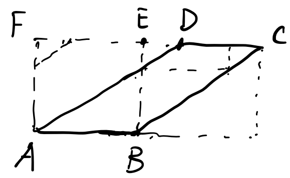


$\text{Area} = a \cdot b$

parallelogram:



$\text{Area} = b \cdot h$



$$|\text{▱}_{ABCD}| = |\text{▱}_{ABCF}| - |\triangle_{ADF}|$$

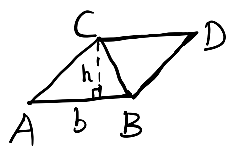
$$= |\text{▱}_{ABCF}| - |\triangle_{BCE}|$$

$$= |\square_{ABEF}| = |AB| \cdot |BE|$$

base · height

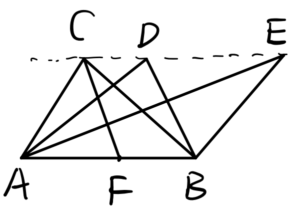
$$\triangle_{ADF} \overset{SAS}{\cong} \triangle_{BCE}$$

Triangle:



$$|\triangle_{ABC}| = \frac{1}{2}|\text{▱}_{ABDC}| = \frac{1}{2} \cdot b \cdot h$$

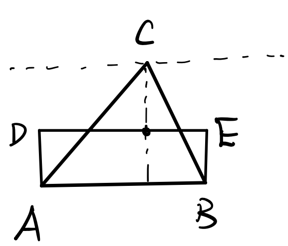
$\rightsquigarrow$ •



$$|\triangle_{ABC}| = |\triangle_{ABD}| = |\triangle_{ABE}|$$

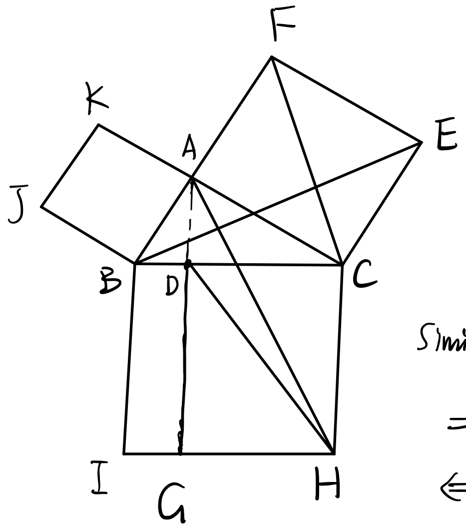
$$\frac{|\triangle_{AFC}|}{|\triangle_{ABC}|} = \frac{|AF|}{|AB|}$$

•



$$|\triangle_{ABC}| = |\square_{ABED}|$$

Pythagorean Thm



$$|\triangle_{DGH}| = |\triangle_{CHA}| \overset{(SAS)}{=} |\triangle_{CBE}|$$
$$|\triangle_{CBE}| = |\triangle_{CFE}|$$

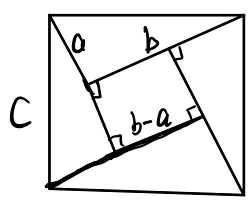
$\Rightarrow |\square_{CDGH}| = |\square_{ACEF}|$

Similarly $|\square_{BDGI}| = |\square_{ABJK}|$

$\Rightarrow |\square_{ABJK}| + |\square_{ACEF}| = |\square_{BCHI}|$

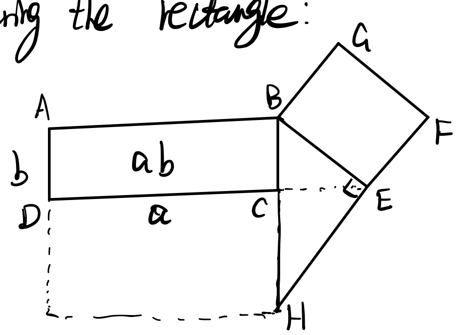
$\Leftrightarrow \quad |AB|^2 + |AC|^2 = |BC|^2.$

Another quick proof:



$$(b-a)^2 + 4 \cdot \tfrac{1}{2} ab = c^2$$
$$(b-a)^2 + 4 \cdot \tfrac{1}{2} ab = a^2 + b^2$$

- squaring the rectangle:



$|\square_{ABCD}| = |\square_{BEFG}|$

$ab = |BE|^2$

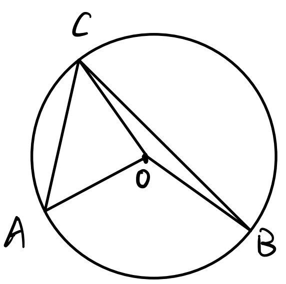
Another proof: $\triangle BEH \sim \triangle BCE$

$\Rightarrow \dfrac{|BH|}{|BE|} = \dfrac{|BE|}{|BC|} \Leftrightarrow |BE|^2 = |BH| \cdot |BC|.$

$\rightsquigarrow$ squaring the triangle $\rightsquigarrow$ squaring any n-polygon

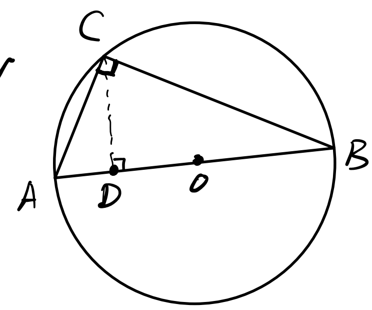
squaring sum of triangles

Recall:



$$2\angle ACB = \Delta_{AOB}$$

In particular

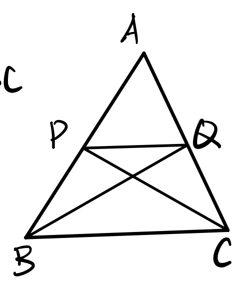


$$\angle ACB = \frac{\pi}{2}$$

$$|AC|^2 = |AD| \cdot |AB|.$$

- Proof of Thales Thm ($\Rightarrow$ similar triangles have proportional sides)

$PQ // BC$



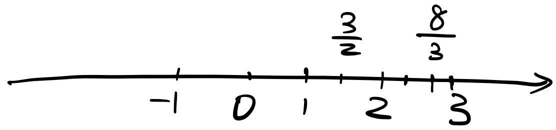
$$\frac{|AP|}{|AB|} = \frac{|\Delta_{APC}|}{|\Delta_{ABC}|} = \frac{|\Delta_{AQB}|}{|ABC|} = \frac{|AQ|}{|AC|}$$

$$|\Delta_{PQB}| = |\Delta_{PQC}|.$$

$\blacksquare$

Coordinates.

real line:



$\mathbb{Q} = \left\{ \frac{m}{n} : \ m, n \in \mathbb{Z}, \ n \neq 0 \right\}. \underset{\text{dense}}{\subset} \mathbb{R}$

Irrational numbers: $\mathbb{R} \setminus \mathbb{Q}$.

Ex: $\sqrt{2}$ is an irrational number.

Pf: Suppose $\sqrt{2} = \frac{m}{n}$ with m, n relatively prime.

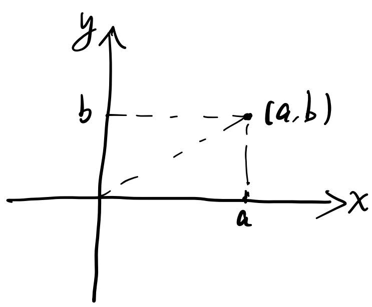
Then $\sqrt{2} \cdot n = m \Rightarrow 2n^2 = m^2 \underset{\uparrow}{\Rightarrow} m$ is even, $m = 2m_1$

unique factorization of integers.

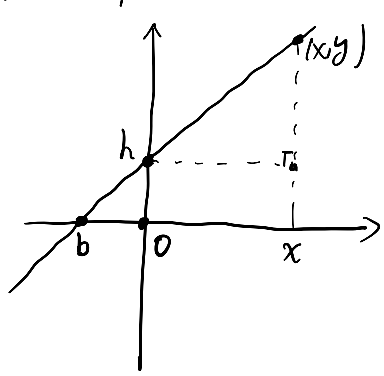
$\Rightarrow 2n^2 = (2m_1)^2 = 4m_1^2 \Rightarrow n^2 = 2m_1^2 \Rightarrow n$ is even.

contradicting m, n relatively prime. ∎

Real plane $\mathbb{R}^2$



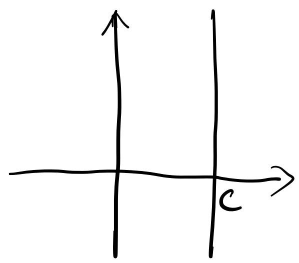
equation for lines



Similar right triangles:

$$\frac{y-h}{x} = \frac{h}{-b} = k \ \text{(slope)}$$

$$\Rightarrow \ y = kx + h$$

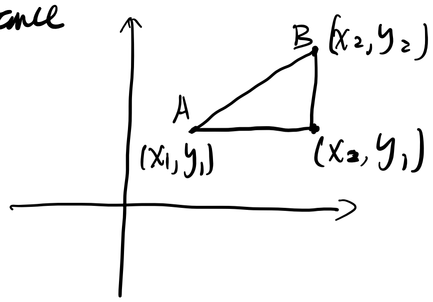


infinite slope: $x = C$.

linear equation $\rightsquigarrow$ lines

$$\alpha x + \beta y + \gamma = 0 \longleftrightarrow \begin{cases} y = -\frac{\alpha}{\beta}x - \frac{\gamma}{\beta} & \beta \neq 0 \\ x = -\frac{\gamma}{\alpha} & \beta = 0, \alpha \neq 0 \end{cases}$$

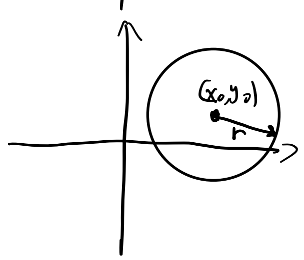
distance



$$|AB|^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$\Rightarrow d(A,B) = |AB| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

equation for circles:



$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

Intersection of lines:

Cramer's formula:

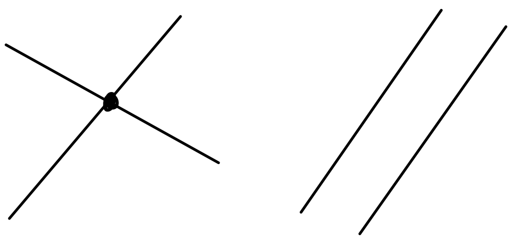
$$\begin{cases} \alpha_1 x + \beta_1 y = \gamma_1 \\ \alpha_2 x + \beta_2 y = \gamma_2 \end{cases} \Rightarrow x = \frac{\begin{vmatrix} \gamma_1 & \beta_1 \\ \gamma_2 & \beta_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}} = \frac{\gamma_1\beta_2 - \gamma_2\beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1}$$

$$\parallel$$

$$\frac{\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2}}{\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}}$$

$$\Downarrow$$

$$y = \frac{\begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}} = \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1\beta_2 - \alpha_2\beta_1}$$



rational functions of slopes and intercepts.

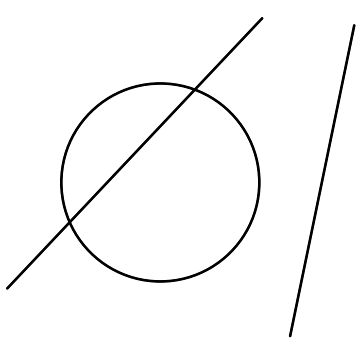
Intersection of line with circle:

$$\begin{cases} \alpha x + \beta y = \gamma \quad \Rightarrow \quad y = kx + h \\ (x - x_0)^2 + (y - y_0)^2 = r^2 \end{cases} \Rightarrow (x - x_0)^2 + (kx + h - y_0)^2 = r^2$$

$$\downarrow$$

$$Ax^2 + Bx + C = 0$$

$$A = 1 + k^2, \quad B = -2x_0 + 2k\cdot(h - y_0), \quad C = x_0^2 + (h - y_0)^2 - r^2.$$

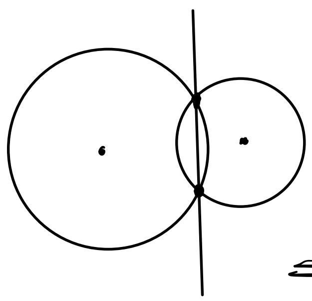


$$\Downarrow$$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

quadratic function of coefficients

- Intersection of circles:



$$(x-x_1)^2+(y-y_1)^2=r_1^2$$

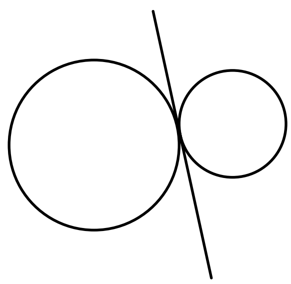
$$(x-x_2)^2+(y-y_2)^2=r_2^2$$

$\Rightarrow$ subtract

$$2(-x_1+x_2)\cdot x+2(-y_1+y_2)\cdot y = r_1^2-r_2^2-x_1^2-y_1^2+x_2^2+y_2^2$$

$\uparrow$

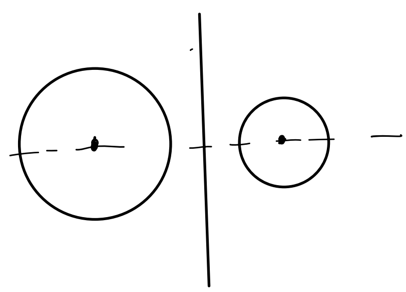
line passing throug intersections or tangent.

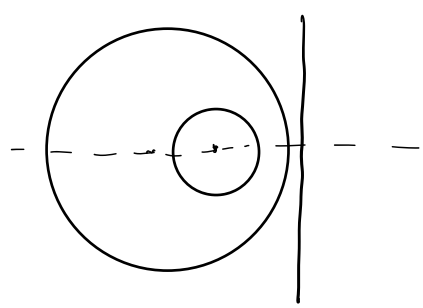


$\Downarrow$

coordinates of intersection points are solutions to quadratic equations.

<u>Q</u>: What is the line when circles are disjoint?

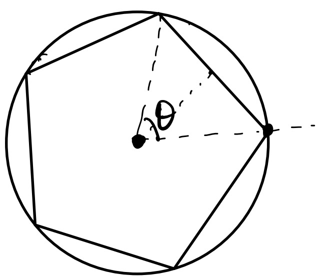




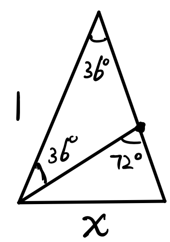
Algebraic criterion for constructibility:

A point is constructible (via straightedge/compass) starting from 0 and if and only if its coordinates are obtainable from the number 1 by the operations $+, -, \times, \div$ and $\sqrt{\cdot}$.

<u>Ex</u>: regular 5-gon:



$$\theta = \frac{2\pi}{5} = 72^\circ$$



$$\frac{1}{x} = \frac{x}{1-x}$$

$$\Rightarrow x^2 + x - 1 = 0$$

$$\Rightarrow x = \frac{-1+\sqrt{5}}{2}$$

$\Rightarrow \cos(72^\circ) = \frac{x}{2} = \frac{-1+\sqrt{5}}{4}$ is constructible

$\Rightarrow$ regular 5-gon is constructible.

Recall: regular n-polygon is constructible

$$\Updownarrow$$

$\cos\left(\frac{2\pi}{n}\right)$ is constructible

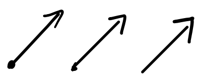
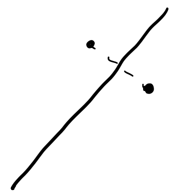
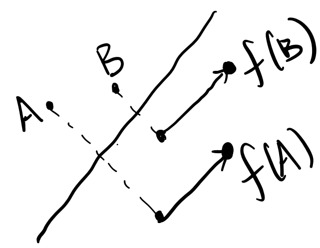
$$\Updownarrow$$

$n = 2^k \cdot p_1 \cdots p_r$, $p_i = 2^{2^{m_i}} + 1$ is a Fermat prime

$3, 5, 17, 257, 65537, ?$

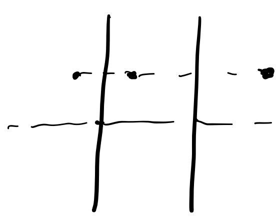
Isometries: A transformation (map) is an isometry if
$f: \mathbb{R}^2 \to \mathbb{R}^2$

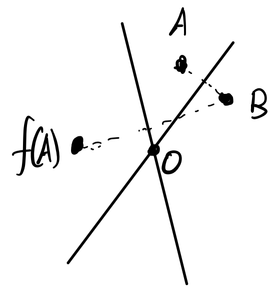
$|f(A)f(B)| = |AB|$ for any two points A, B.

translation	rotation	reflection	glided reflection
			
$(x,y) \mapsto (x+c_1, y+c_2)$			

<u>Thm</u>: 1. Any isometry of $\mathbb{R}^2$ is a combination of one, two or three reflections.

2. Any isometry is one of the above 4 types.





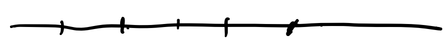
two reflections:

$|AO| = |BO| = |f(A)O|$

$x \quad a \quad 2a-x \quad b \quad 2b-(2a-x)$

$\qquad\qquad\qquad\qquad\qquad\quad \parallel$

translation $\qquad 2(b-a)+x$



$\theta \quad \alpha \quad 2\alpha-\theta \quad \beta \quad 2(\beta-\alpha)+\theta$

translation of angles $\Rightarrow$ rotation