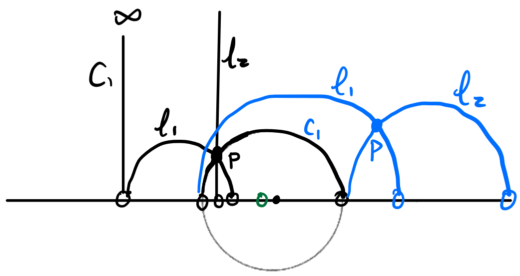
Non-Euclidean Geometry : Upper half plane

"lines" : vertical lines or semi-circles

Each line has two "ends" = "infinite points"



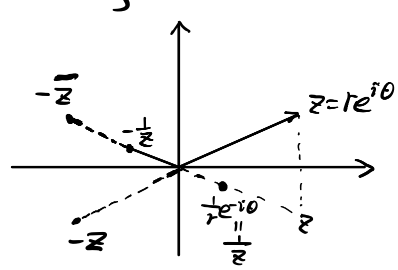
Given any line C_1 with two ends Q_1, Q_2

For any $P \notin C_1$, there are infinitely many lines that pass through P and that don't intersect C_1. Among these lines, there is exactly one line ℓ_1 that is asymptotic to Q_1 and one line l_2 that is asymptotic to Q_2.

$$\mathrm{Isom}^+(U) = \left\{ f(z) = \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R},\ ad-bc>0 \right\}$$

$$\frac{az+b}{cz+d} = \frac{a}{c} + (ad-bc)\cdot\frac{-1}{c^2(z+\frac{d}{c})}$$

$$= \left\langle z+l,\ \underset{k>0}{k\cdot z},\ -\frac{1}{z} \right\rangle$$



$$\mathrm{Isom}(U) = \left\{ \frac{az+b}{cz+d},\ ad-bc>0\ ;\ \frac{a\bar{z}+b}{c\bar{z}+d},\ ad-bc<0 \right\}.$$

$$= \left\langle z+l,\ \underset{k>0}{k\cdot z},\ -\frac{1}{z},\ -\bar{z} \right\rangle$$

reflection across y-axis.

- $f \in \mathrm{Isom}(U)$ transforms "lines" to "lines".

Verify for $f(z) = -\frac{1}{z}$ and the vertical line $x = a \iff z + \bar{z} = 2a$.

$$x = \frac{1}{2}\mathrm{Re}\, z$$

$$\left.\begin{array}{l} w = -\frac{1}{z} \iff z = -\frac{1}{w} \\ z + \bar{z} = 2a \end{array}\right\} \Rightarrow -\frac{1}{w} - \overline{\frac{1}{w}} = 2a$$

$$-\frac{1}{w} - \overline{\frac{1}{w}} = -\left(\frac{1}{w} + \frac{1}{\bar{w}}\right) = -\frac{w + \bar{w}}{|w|^2}$$

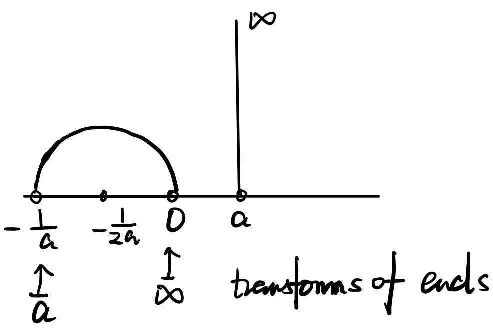
If $a = 0$, then $w + \bar{w} = 0 \iff \mathrm{Re}\, w = x = 0$.

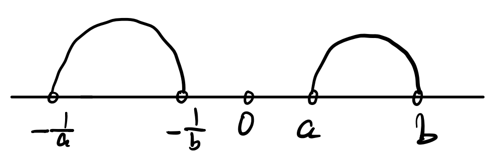
If $a \neq 0$, then $|w|^2 + \frac{1}{2a} w + \frac{1}{2a} \bar{w} = 0$

$$\Updownarrow$$

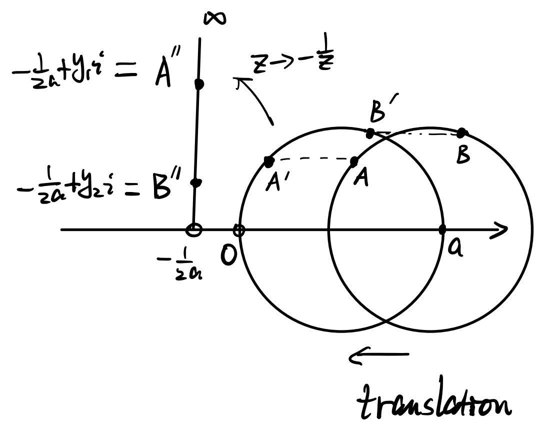
$$\left|w - \left(-\frac{1}{2a}\right)\right|^2 = \frac{1}{4a^2}$$

circle centered at $\left(-\frac{1}{2a}\right)$ with radius $\frac{1}{2a}$.





Define Distance that is preserved under $\mathrm{Isom}(U)$.



$$\begin{aligned} \mathrm{dist}(A,B) &= \mathrm{dist}(A',B') \\ &= \mathrm{dist}(A'',B'') \\ &= \mathrm{dist}(y_1 i, y_2 i) \end{aligned}$$

want $\mathrm{dist}(y_1 i, y_2 i) = \mathrm{dist}(k\cdot y_1 i, k y_2 \cdot i) \quad \forall k>0$

$\rightsquigarrow \quad \mathrm{dist}(y_1 i, y_2 i) = f\left(\frac{y_2}{y_1}\right) = f\left(\frac{y_1}{y_2}\right) = \mathrm{dist}(y_2 i, y_1 i)$

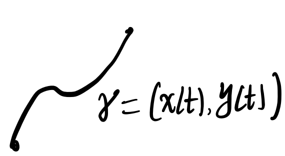
want $\quad \mathrm{dist}(y_1 i, y_3 i) = \mathrm{dist}(y_1 i, y_2 i) + \mathrm{dist}(y_2 i, y_3 i)$

$$f\left(\frac{y_3}{y_1}\right) = f\left(\frac{y_2}{y_1}\right) + f\left(\frac{y_3}{y_2}\right)$$

$\rightsquigarrow \quad f = |\log| = |\ln|$ (up to a constant)

So $\quad \mathrm{dist}(y_1 i, y_2 i) = \left|\log \frac{y_2}{y_1}\right|$.

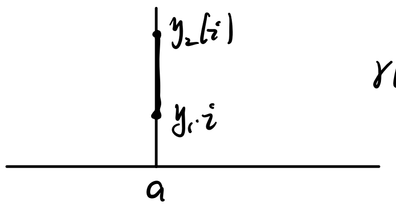
Another way to define using differential geometry:



Euclidean length: $\quad \mathcal{L}(\gamma) = \int_{t_1}^{t_2} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt.$

Non-Euclidean length: $L(\gamma) = \int_{t_1}^{t_2} \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)}\, dt.$

Example:

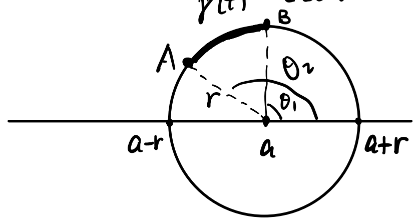


$\gamma(t) = (a, t)$, $y_1 < t < y_2$

$$\Rightarrow L(\gamma) = \int_{y_1}^{y_2} \frac{\sqrt{0^2+1^2}}{t}\, dt = \int_{y_1}^{y_2} \frac{dt}{t} = \log t \Big|_{y_1}^{y_2} = \log \frac{y_2}{y_1}.$$

Example:

$\gamma(t) = (a + r\cos t,\ r\sin t)$. $\theta_1 \le t \le \theta_2$



$\dot{\gamma}(t) = (-r\cdot\sin t,\ r\cos t)$

$|\dot{\gamma}(t)| = r$

$$L(\gamma) = \int_{\theta_1}^{\theta_2} \frac{r}{r\sin t}\, dt = \int_{\theta_1}^{\theta_2} \frac{dt}{\sin t} = \int_{\theta_1}^{\theta_2} \frac{\sin t\, dt}{\sin^2 t}$$

$$= \int_{\theta_1}^{\theta_2} \frac{-d\cos t}{1-\cos^2 t} \overset{u=\cos t}{=\!=} \int_{\cos\theta_1}^{\cos\theta_2} \frac{-du}{1-u^2}$$

$$\int \frac{du}{1-u^2} = \int \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right) du = \frac{1}{2}\log\frac{1+u}{1-u}$$

$$= -\frac{1}{2}\log\frac{1+u}{1-u}\Big|_{\cos\theta_1}^{\cos\theta_2} = -\frac{1}{2}\log\frac{1+\cos\theta}{1-\cos\theta}\Big|_{\theta_1}^{\theta_2}$$

$$= -\frac{1}{2}\log\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\Big|_{\theta_1}^{\theta_2} = \log\tan\frac{\theta_2}{2} - \log\tan\frac{\theta_1}{2} = \left|\log\frac{\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2}}\right|$$

Verify: Use transformation: $z \longmapsto -\dfrac{1}{z-(a+r)}$

$a + r\cos\theta + i \cdot r\sin\theta = a + r \cdot e^{i\theta}$ is mapped to

$$-\frac{1}{a+r\cdot e^{i\theta}-(a+r)} = -\frac{1}{re^{i\theta}-r} = -\frac{1}{r}\cdot\frac{1}{(\cos\theta-1)+i\cdot\sin\theta}$$

$$= \frac{1}{r}\cdot\frac{1}{2\cos^2\frac{\theta}{2} - i\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{1}{2r\cdot\cos\frac{\theta}{2}}\cdot\left(\cos\frac{\theta}{2} + i\cdot\sin\frac{\theta}{2}\right)$$

$$= \frac{1}{2r} + \frac{1}{2r} i \cdot \tan\frac{\theta}{2}$$

$$\Rightarrow \operatorname{dist}(A,B) = \operatorname{dist}\left(\frac{1}{2r} i \cdot \tan\frac{\theta_1}{2}, \frac{1}{2r} i \cdot \tan\frac{\theta_2}{2}\right) = \left|\log\frac{\tan\frac{\theta_1}{2}}{\tan\frac{\theta_2}{2}}\right|.$$

- More general length $\int_{t_1}^{t_2} F(x,y)|\dot{\gamma}(t)|\,dt$ $\rightsquigarrow$ Riemannian geometry

  $F \equiv 1$: Euclidean Geometry

  $F = \frac{1}{y}$: Non-Euclidean Geometry.