Claim: Rotation $f_q = f_{q'}$, $q, q' \in S^3 \Leftrightarrow q' = \pm q$

Proof: $f_q = f_{q'} \Leftrightarrow q \cdot p \cdot q^{-1} = q' \cdot p \cdot q'^{-1} \quad \forall p \in S^2$

$$\Leftrightarrow \underbrace{(q^{-1} q')}_{= q_1} p = p (q^{-1} q') \quad \forall p \in S^2$$

$q_1 = a + bi + cj + dk \qquad \underbrace{q_1 i}_{= ai - b - ck + dj} = \underbrace{i q_1}_{= ai - b + ck - dj} \Rightarrow c = d = 0$

Similarly $q_1 j = j q_1 \Rightarrow b = d = 0$.

So $q_1 = a \in \mathbb{R}$, $|q_1| = |q|^{-1} \cdot |q'| = 1 \Rightarrow q_1 = \pm 1$. ∎

So each rotation corespond to exactly 2 quaternions:

rotation around axis $li + mj + nk$ by angle θ

$\|$

rotation around axis $(li + mj + nk)$ by angle $2\pi + \theta$

$\rightsquigarrow \quad q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li + mj + nk)$ and

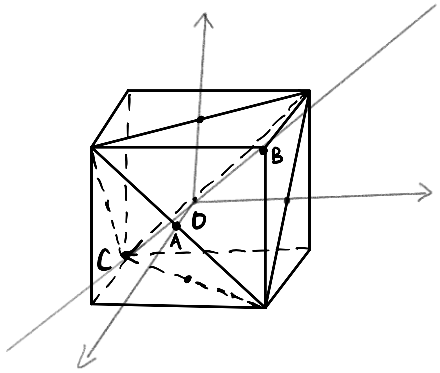
$$q' = \cos\frac{2\pi+\theta}{2} + \sin\frac{2\pi+\theta}{2}(li + mj + nk)$$

$$\| $$

$$-\cos\frac{\theta}{2} - \sin\frac{\theta}{2}(li + mj + nk) = -q$$

$$\boxed{f_q = f_{-q}}$$

Rotations of tetrahedron



Rotation around x-axis by angle π

$$\updownarrow$$

$$\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cdot(1\cdot i + 0\cdot j + 0\cdot k)$$

$$\shortparallel$$

$$i \rightsquigarrow \pm i.$$

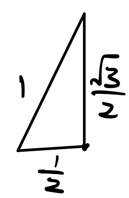
Rotations around y-axis $\leftrightarrow j\ ,\ -j$

z-axis $\leftrightarrow k\ ,\ -k$

Ration around the axis BC by angle $\frac{2\pi}{3}\ ,\ \frac{4\pi}{3}$

axis direction $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)-\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)=(1,1,1).$

$$\rightsquigarrow li + mj + nk = \frac{1}{\sqrt{3}}(i+j+k)$$



$$\rightsquigarrow \cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot\frac{1}{\sqrt{3}}(i+j+k) = \frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}}(i+j+k)$$

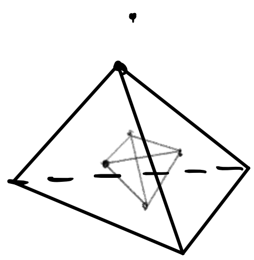
$$= \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$$

$$\rightsquigarrow \left\{\pm 1, \pm i, \pm j, \pm k, \pm\frac{1}{2}\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k\right\} = V$$

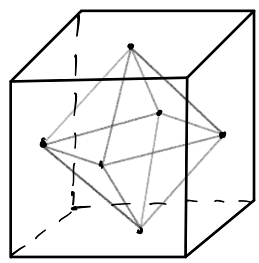
24 quaternions representing 12 rotations of tetrahedron.

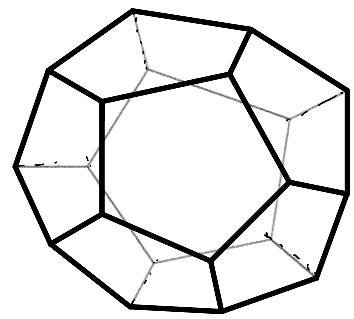
$V \subset \mathbb{R}^4$: vertex of the 24-cell.

- Regular polytopes = Platonic solids.

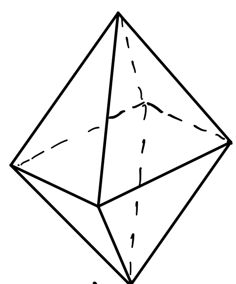


tetrahedron
(self dual)

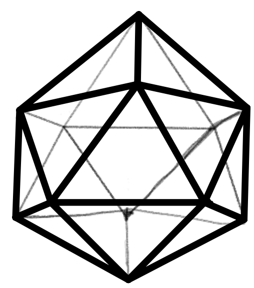




Dodecahedron



Octahedron.



Icosahedron

	tetrahedron	cube	Octahedron	Dodecahedron	Icosahedron
v	4	8	6	20	12
e	6	12	12	30	30
f	4	6	8	12	20

Euler formula: $v - e + f = 2 = \chi(S^2)$ (Euler characteristic)

- Determine all possible regular polytopes:

Assume
- Each face is a regular k-polygon
- each vertex has d edges passing through it

$$\Rightarrow \quad \frac{(k-2)\pi}{k} \cdot d < 2\pi \iff d < \frac{2k}{k-2} \qquad k \geq 3, \quad d \geq 3$$

$k=3$: $\quad d < \frac{2 \cdot 3}{3-2} = 6 \Rightarrow d = 3, 4, 5$

$k=4$: $\quad d < \frac{2 \cdot 4}{4-2} = 4 \Rightarrow d = 3$

$k=5$: $\quad d < \frac{2 \cdot 5}{5-2} = \frac{10}{3} \Rightarrow d = 3$

$k \geq 6$. $\quad \frac{2k}{k-2} = 2 + \frac{4}{k-2} \leq 3 \Rightarrow d \leq 2$

So all possibilities for (k, d):

$(3,3)$, $(3,4)$, $(3,5)$
$(4,3)$
$(5,3)$

Exercise: • Express e and f as functions of v, k, d

• Use Euler's formula to calculate v, e, f to recover the following table:

k	3	3	3	4	5
d	3	4	5	3	3
v	4	6	12	8	20
e	6	12	30	12	30
f	4	8	20	6	12
	↑ tetrahedron	↑ octahedron	↑ icosahedron	↑ cube	↑ dodecahedron.

high dimensions: There are always

tetrahedron, cube, octahedron in any dimension

Exceptional regular polytopes only in

dim 3: Dodecahedron, Icosahedron

dim 4: 24-cell, 600-cell

24 vertices

$\rightsquigarrow$ rotations of tetrahedron

(self dual)

120 vertices

$\rightsquigarrow$ rotations of icosahedron

$\updownarrow$ dual

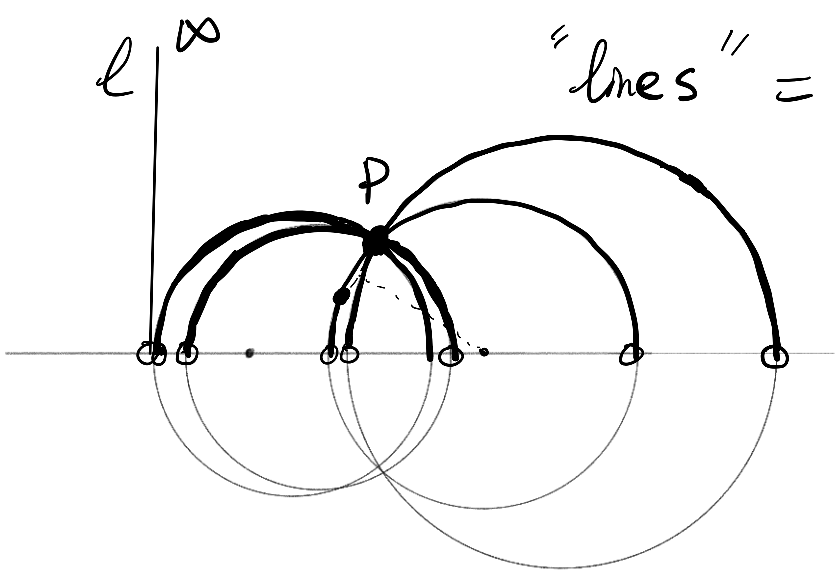
120-cell

600 vertices

- Non-Euclidean Geometry.

Upper half plane

$U = \{ z = x + iy,\ y > 0 \}$



"lines" = vertical lines; semi-circles that are perpendicular to the x-axis

through $P \notin \ell$, there are infinitely many "lines" passing throug P that does not intersect ℓ.

Infinity line of $U = \mathbb{R} \cup \{\infty\} = \mathbb{R}P^1$.

linear fractional transformation of $\mathbb{R}P^1$

$$f(x) = \frac{ax+b}{cx+d} \qquad x \in \mathbb{R}$$

$$= \frac{a(x+\frac{d}{c})}{cx+d} + \frac{b-\frac{ad}{c}}{cx+d} = \frac{a}{c} - \frac{ad-bc}{c^2(x+\frac{d}{c})}$$

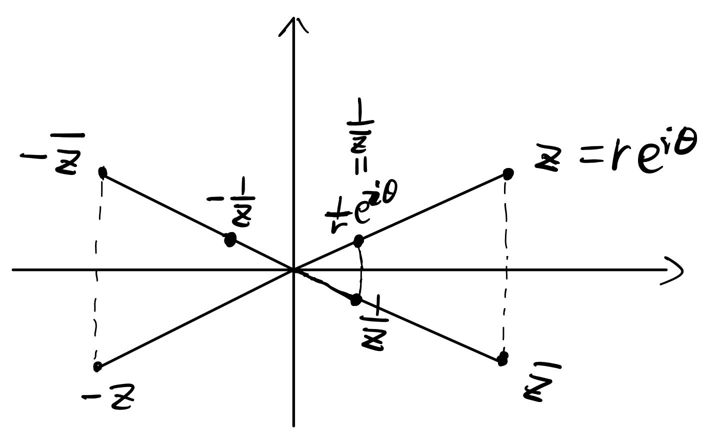
basic operation: extend to transformation of U

$$x \longmapsto x+l \quad \rightsquigarrow \quad z \longmapsto z+l$$

$$x \longmapsto kx \quad \rightsquigarrow \quad z \longmapsto k\cdot z, \quad k>0$$

$$x \longmapsto -x \quad \rightsquigarrow \quad z \longmapsto -\bar{z}.$$

$$x \longmapsto \frac{1}{x} \quad \rightsquigarrow \quad z \longmapsto -\frac{1}{z}$$



$$\frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

$$-\frac{1}{z} = \frac{1}{r}e^{i\theta}$$

General linear fractional transformation

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c^2(z+\frac{d}{c})}$$ preserves the upper half plane

if and only if $ad-bc>0$.

$$\mathrm{Isom}^+(U) = \left\{ \frac{az+b}{cz+d} : \begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc>0 \end{array} \right\}$$

$z \mapsto -\bar{z}$ reverses the orientation.

- Isometries of U transforms "lines" to "lines"

Just need to verify this for basis transformations:

$$z \mapsto z+l, \quad z \mapsto k\cdot z, \quad z \mapsto -\frac{1}{z}, \quad z \mapsto -\bar{z}$$

Use equation for lines:

vertical line: $\mathrm{Re}\, z = a$

semi-circle: $|z-a|^2 = r^2$.