Straightedge/Compass construction

Euclid: Elements (300 BCE)

- 
- extend line indefinitely
- compass $\rightsquigarrow$ circle.

- Construction of equilateral triangle.

<u>Q</u>: regular n-polygons by SC construction.

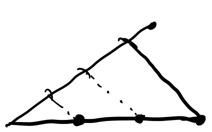
<u>A</u>: only when $n = 2^k \cdot p_1 \cdots p_\ell$ where $p_i = 2^{2^{m_i}} + 1$ is a (Fermat) prime number.

m	0	1	2	3	4	
$2^{2^m}+1$	3	5	17	257	65537	?
		(Euclid)	(Gauss)			

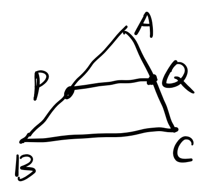
<u>Conj</u>: No other Fermat primes.

- bisect a line segment
- construct perpendicular lines, parallel lines
- divide a line segment into n equal parts for any n.

$n=3$:



Thales Thm:

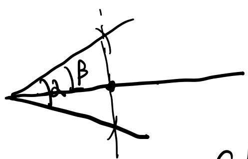


$PQ \,//\, BC$

$\Downarrow$

$$\frac{|AP|}{|AB|} = \frac{|AQ|}{|AC|}.$$

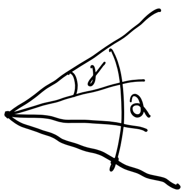
- bisect an angle $\Leftrightarrow$ bisect a line segment



$$\cos(\alpha) = \cos(2\beta) = 2\cos^2(\beta) - 1$$

solve for β $\Leftrightarrow$ solve for $\cos(\beta)$ $\Leftrightarrow$ solve a quadratic eq.

- Q: trisect an angle ($\not\Leftrightarrow$ trisect a line segment)



$$\begin{aligned}\cos(\alpha) = \cos(3\gamma) &= \cos(2\gamma)\cos(\gamma) - \sin(2\gamma)\sin(\gamma)\\ &= (2\cos^2(\gamma)-1)\cos(\gamma) - 2\sin(\gamma)\cos(\gamma)\sin(\gamma)\\ &= 2\cos^3\gamma - \cos\gamma - 2\cdot(1-\cos^2\gamma)\cos\gamma\\ &= 4\cos^3\gamma - 3\cos\gamma.\end{aligned}$$

solve for γ $\Leftrightarrow$ solve for $\cos(\gamma)$

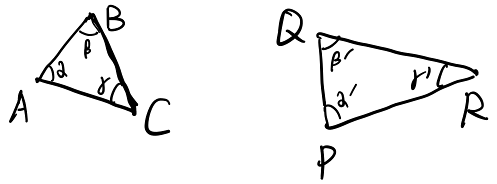
$\Leftrightarrow$ solve a cubic equation

<u>Fact</u>: Straightedge/compass construction can only solve quadratic equation.

To solve cubic equation, can use either
- markable ruler (Neusis construction)
- or origami

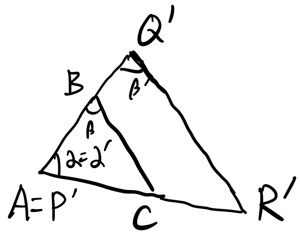
- Similar triangles:

$\Delta_{ABC} \sim \Delta_{PQR}$ if
$$\overset{\angle A}{\alpha} = \overset{\angle P}{\alpha'}, \quad \beta = \beta', \quad \gamma = \gamma'.$$



<u>Proposition</u>: $\Delta_{ABC} \sim \Delta_{PQR} \implies \dfrac{|AB|}{|PQ|} = \dfrac{|BC|}{|QR|} = \dfrac{|AC|}{|PR|}$.

<u>Proof</u>: • "Move" $\Delta_{PQR} \longrightarrow \Delta_{P'Q'R'}$



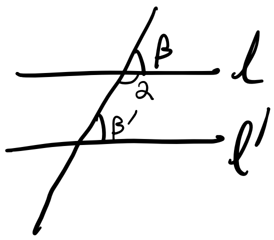
- $\beta = \beta' \implies Q'R' \parallel BC$
- Thales Thm $\implies \dfrac{|AB|}{|AQ'|} = \dfrac{|AC|}{|AR'|}$

$$\overset{\shortparallel}{\dfrac{|AB|}{|PQ|}} \qquad \overset{\shortparallel}{\dfrac{|AC|}{|PR|}}$$

Similarly $\dfrac{|AB|}{|PQ|} = \dfrac{|BC|}{|QR|}$. ∎

<u>Rmk</u>: · Use the operation of translation & rotation

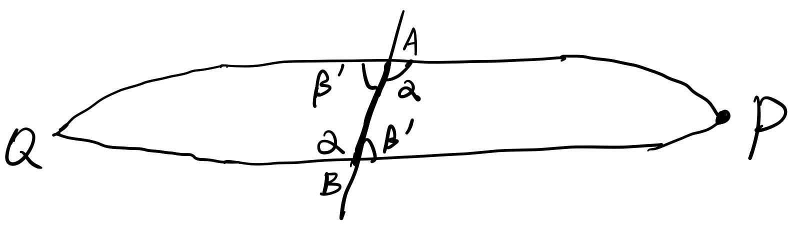
· use a criterion for being parallel.
(the following prop.)

Prop: 

$\beta = \beta' \Rightarrow l \parallel l'$

Equivalently $\alpha + \beta' = \pi \Rightarrow l \parallel l'$.

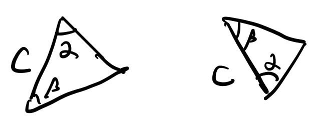
Proof: proof by contradiction. Suppose $l \cap l' = P$



Then there are congruent triangles $\triangle ABP \cong \triangle BAQ$ by the <u>ASA axiom</u>.

Then there are two lines l and l' passing through P and Q. A contradiction.

- Three axioms:

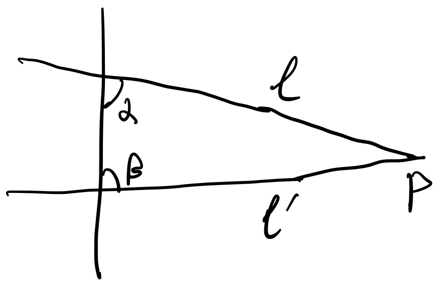
ASA (Angle-Side-Angle 

)

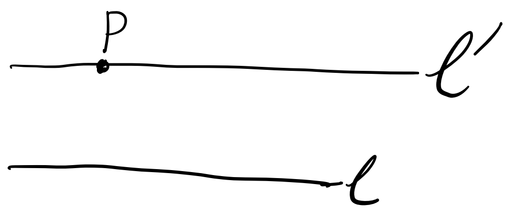
SAS

SSS

- Parallel Axiom: $\alpha+\beta<\pi \Longrightarrow l \cap l' = P$ on the right side.



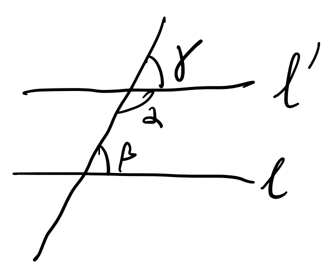
Playfair Axiom:



For any line l and $P \notin l$, there exists a unique line l' through P that does not meet l.

Parallel Axiom $\Longleftrightarrow$ Playfair Axiom. (P)

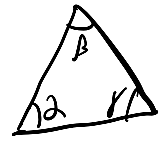
(P) implies:



$$l' \parallel l \Longrightarrow \alpha+\beta=\pi$$
$$\Updownarrow \qquad (\alpha+\gamma=\pi)$$
$$\beta=\gamma$$

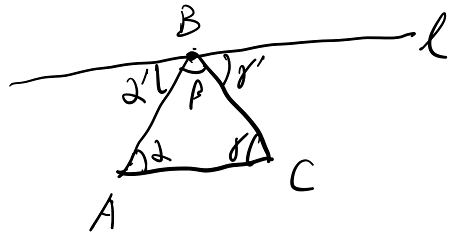
So we get $l' \parallel l \Longleftrightarrow \alpha+\beta=\pi \Longleftrightarrow \beta=\gamma$.

"$\Longrightarrow$" uses (P).

· **Proposition:** 

$\alpha+\beta+\gamma=\pi$.

Proof:

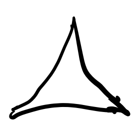
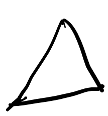
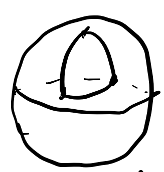


construct $l \parallel AC$, $B \in l$. Then $\alpha+\beta+\gamma \overset{\text{use (P)}}{=} \alpha'+\beta+\gamma'=\pi$. ∎

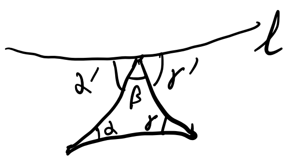
· Equivalent result

$\pi-\gamma=\alpha+\beta$

· There are geometries for which (P) does NOT hold.

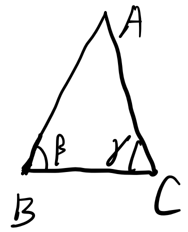
hyperbolic	flat	spherical
		

↑ "lines" are great circles (geodesics)



$\alpha+\beta+\gamma<\alpha'+\beta+\gamma'=\pi$.

- **Prop**: (Isoceles Triangle Thm).



$|AB| = |AC| \Rightarrow \beta = \gamma$

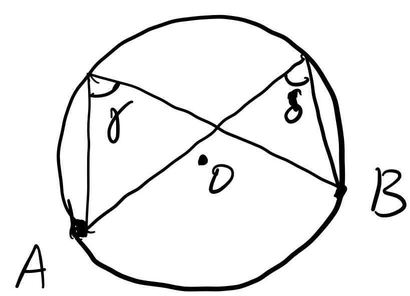
Proof: $|AB| = |AC|$, $|BC| = |CB|$, $|CA| = |BA|$ $\xRightarrow{(SSS)}$ $\triangle ABC \cong \triangle ACB$

$$\Rightarrow \underbrace{\angle ABC}_{\beta} = \underbrace{\angle ACB}_{\gamma}. \quad \blacksquare$$

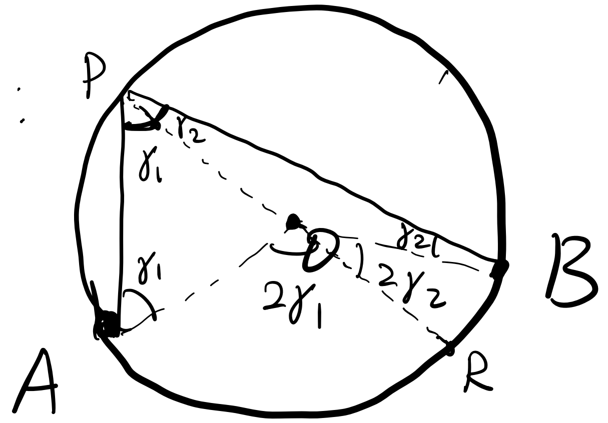
Application of Isoceles Thm:

- **Prop** (Angles in a circle)

$\gamma = \delta$



Proof:

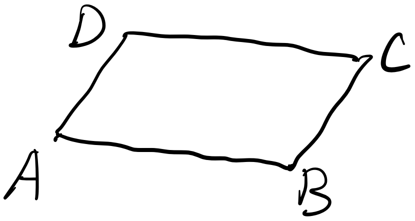


$\triangle OAP$ and $\triangle OPB$ are isosceles triangles.

$$\left.\begin{aligned} |OP|=|OA| &\Rightarrow 2\gamma_1 = \angle AOR \\ |O\ |=|OB| &\Longrightarrow 2\gamma_2 = \angle BOR \end{aligned}\right\} \Rightarrow 2\gamma = 2\gamma_1 + 2\gamma_2 = \angle AOB$$

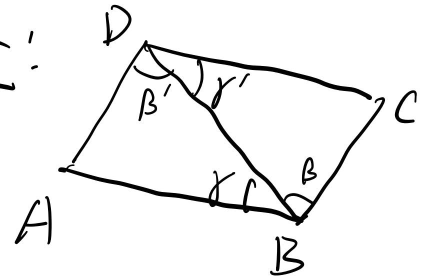
Similarly $2\delta = \angle AOB$. So $2\gamma = 2\delta \Rightarrow \gamma = \delta$ ∎

Prop: (Parallelogram side Thm)



$$\begin{matrix} AB /\!/ DC \\ AD /\!/ BC \end{matrix} \Longrightarrow \begin{matrix} |AB| = |CD| \\ |AD| = |BC| \end{matrix}$$

Proof:



Use (P): $\beta = \beta'$ and $\gamma = \gamma'$

Use (ASA): $\triangle ABD \cong \triangle CDB$

So $|AB| = |CD|$, $|AD| = |BC|$. ∎