1. (i) Calculate the linear fractional transformation f of $\mathbb{RP}^1$ that satisfies $f(0)=1$, $f(1)=2$, $f(\infty)=3$.

(ii) For the f from (i), calculate the cross ratio:

$$[f(2), f(3), f(5), f(7)]$$

(i) Let $f(x)=\dfrac{ax+b}{cx+d}$ Then

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$$1=f(0)=\frac{b}{d},\quad 2=f(1)=\frac{a+b}{c+d},\quad 3=f(\infty)=\frac{a}{c}$$

$$\Rightarrow \begin{cases} b=d \\ a+b=2c+2d \\ a=3c \end{cases} \Rightarrow 3c+b=2c+2b \Rightarrow c=b \Rightarrow \begin{cases} a=3c=3b \\ b=b \\ c=b \\ d=b \end{cases}$$

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$$\Rightarrow f(x)=\frac{3bx+b}{bx+b}=\frac{3x+1}{x+1}.$$

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(ii) Because linear fractional transformations preserve cross-ratios.

$$[f(2), f(3), f(5), f(7)] = [2,3,5,7]$$

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$$=\frac{5-2}{5-3}\cdot\frac{7-3}{7-2}=\frac{3\cdot 4}{2\cdot 5}=\frac{6}{5}.$$

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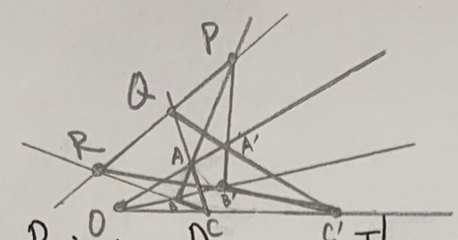
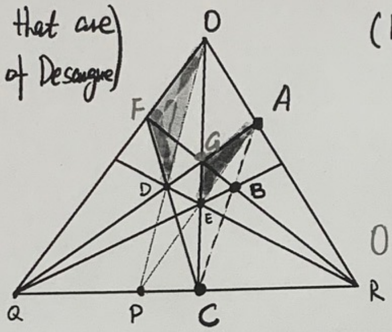


2. (i) State the Projective Desargue Theorem.

(ii) Use the Projective Desargue theorem twice to explane why the 3 points A, B, C in the following picture are on the same line

(Identify the triangles that are used in the application of Desargue)



(Hint: First show $\overline{OD} \cap \overline{AE} = P \in QC$)

Then show $\overline{OE} \cap \overline{AB} \in PR$)

Projective Desargue:

$O = AA' \cap BB' = BB' \cap CC' = AA' \cap CC'$.

$\Updownarrow$

(i) If two triangles ΔABC, $\Delta A'B'C'$ are in perspective, then the intersection points of corresponding sides: $P = AB \cap A'B'$, $Q = AC \cap A'C'$, $R = BC \cap B'C'$ are contained in the same line.

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(ii) ΔOFD and ΔAGE are in perspective from R. So

$OF \cap AG = Q$, $OD \cap AE = P$, $FD \cap GE = C$ are on the same line.

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In other words, $P \in QC =$ line QR

ΔODG and ΔAEB are in perspective from R. So

$OD \cap AE = P$, $OG \cap AB = C'$, $DG \cap EB = Q$ are on the same line

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In other words, $C' \in$ line $PQ =$ line QR.

So $C' =$ line $AB \cap$ line $QR = C$. So A, B, C are on the same line.

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3. For any quaternion $q \in \mathbb{H}$ with $|q|=1$, let

$$f_q: S^2 \to S^2, \ f_q(p) = q \cdot p \cdot q^{-1}$$

denote the associated rotation of the sphere S^2.

(i) Find some q_1 such that

$$f_{q_1}\Big(\underbrace{0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}}_{= v_1}\Big) = \Big(\underbrace{\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0}_{= v_2}\Big).$$

(ii) Calculate $q_2 = q_1 \cdot j$. What is the axis for the rotation f_{q_2} represented by q_2?

(i) Choose axis in the direction of

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{vmatrix} = i \cdot \tfrac{1}{2} - j\left(-\tfrac{1}{2}\right) + k \cdot \left(-\tfrac{1}{2}\right) = \tfrac{1}{2}(i+j-k)$$

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$$\rightsquigarrow li + mj + nk = \tfrac{1}{\sqrt{3}}(i+j-k).$$

angle θ satisfies $\cos\theta = v_1 \cdot v_2 = -\frac{1}{2} \Rightarrow$ $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \frac{1}{2}$, $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \frac{\sqrt{3}}{2}$

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$$\Rightarrow q_1 = \cos\tfrac{\theta}{2} + \sin\tfrac{\theta}{2} \cdot (li + mj + nk)$$

$$= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2} \tfrac{1}{\sqrt{3}}(i+j-k) = \tfrac{1}{2} + \tfrac{1}{2}i + \tfrac{1}{2}j - \tfrac{1}{2}k.$$ 5

(ii) $q_1 j = \left(\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j-\frac{1}{2}k\right)\cdot j$

$= \frac{1}{2}j+\frac{1}{2}k-\frac{1}{2}+\frac{1}{2}i$ $\qquad k\cdot j = -j\cdot k = i.$

$= -\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k. = q_2$ $\quad$ 5

f_{q_2} represents a rotation with axis in the direction of

$+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k = \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}}(i+j+k)$ $\quad$ 5

$\rightsquigarrow \quad l'i+m'j+n'k = \frac{1}{\sqrt{3}}(i+j+k) = u$

rotation angle θ' satisfies $\begin{cases} \cos\frac{\theta'}{2} = -\frac{1}{2} \\ \sin\frac{\theta'}{2} = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow \frac{\theta'}{2} = \pi-\frac{\pi}{3} = \frac{2\pi}{3}$

$\Rightarrow \theta' = \frac{4\pi}{3} = 2\pi - \frac{2\pi}{3}$

$\rightsquigarrow$ rotation around the axis u by angle $\frac{4\pi}{3}$ (or $-\frac{2\pi}{3}$)

$\Updownarrow$

rotation around $(-u)$ by angle $\frac{2\pi}{3}$

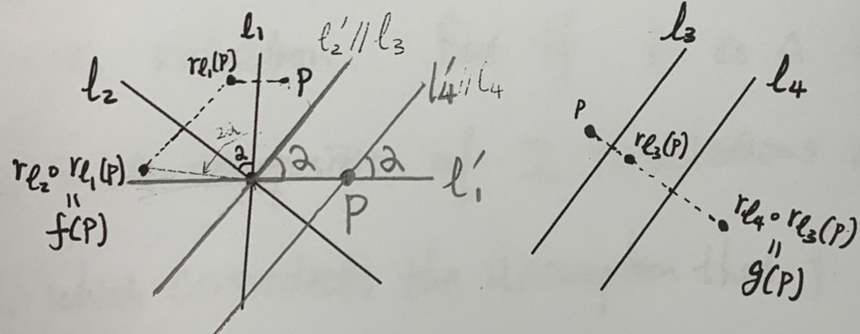
$-q_2 = \frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}}(-i-j-k) = \cos\frac{\frac{2\pi}{3}}{2} + \sin\frac{\frac{2\pi}{3}}{2}\cdot\frac{1}{\sqrt{3}}(-i-j-k).$

4. On the plane $\mathbb{R}^2$, let f be a rotation

Let g be a translation.

(i) Prove $g \circ f$ is a rotation. (You can use the 3 reflection theorem)

(ii) Using the following picture, explain how to use geometric method to find the center and angle for the rotation $g \circ f = (r_{l_4} \circ r_{l_3}) \circ (r_{l_2} \circ r_{l_1})$, where we write $f = r_{l_2} \circ r_{l_1}$ and $g = r_{l_4} \circ r_{l_3}$ as composition of reflections.



(Hint: change the mirrors l_1 to some l_1', and l_4 to some l_4')

(ii) $g \circ f = (r_{l_4} \circ r_{l_3}) \circ (r_{l_2} \circ r_{l_1})$

$= r_{l_4'} \circ r_{l_3'} \circ r_{l_2'} \circ r_{l_1'}$

$\overset{l_2' = l_3'}{=} r_{l_4'} \circ r_{l_1'}$ 5

rotation around $P = l_4' \cap l_1'$ by angle $2a$. 5

- Rotate l_1, l_2 to l_1', l_2' such that $l_2' // l_3$. 5
- Set $l_3' = l_2'$ and translate l_3, l_4 to l_3', l_4'.

Continuation of work:

(i) Both f and g are composition of 2 reflections. So $g \circ f$ is a composition of 4 reflections. In particular $g \circ f$ preserves the orientation (4 is even). By the classification of plane isometries (3 reflection thm), $g \circ f$ can be decomposed as a composition of 2 reflections. So $h = g \circ f$ is either a translation or a rotation. But if h is a translation, then $f = g^{-1} \circ h$ is a composition of 2 translations and is also a translation, which contradicts the assumption that f is a rotation. So $g \circ f$ must be a rotation.

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