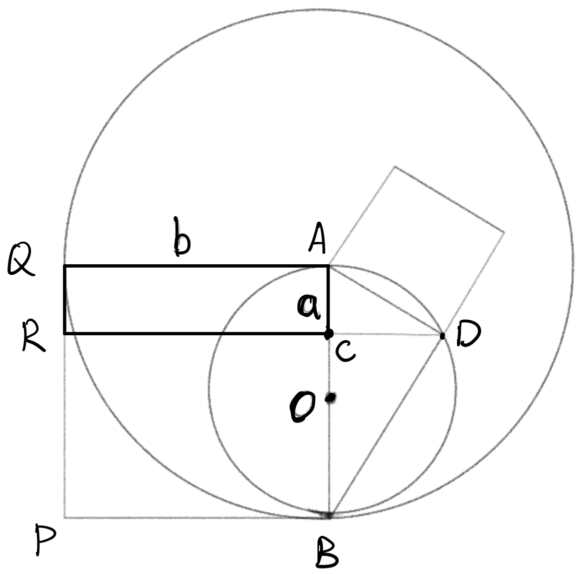
1. Explain how to use straightedge/compass to construct a square that has the same area as the area of a given rectangle.

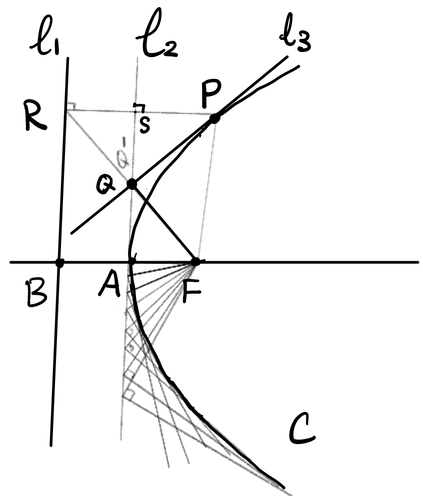


1. Draw a circle with center A and radius $|AQ|$. 4
   Extend AC to intersect this circle at B s.t. $|AQ| = |AB|$
2. Use straightedge/compass to find the middle point O of AB 4
3. Use compass to draw a circle with O the center and $|OA|$ the radius
4. Extend RC to intersect the circle at D 4
5. The square with the side length $|AD|$ satisfies $|AD|^2 = |AC| \cdot |AQ|$. 4

Pf: The two right triangles Δ_{ADB} and Δ_{ACD} are similar. 4

So $\dfrac{|AC|}{|AD|} = \dfrac{|AD|}{|AB|} \Rightarrow |AD|^2 = |AC| \cdot |AB| = |AC| \cdot |AQ|$

2.



Let C be the parabola with the focus F and directrix l_1.

Let l_2 be the line that passes through the vertex A and is parallel to l_1. Let l_3 be any tangent line at $P \in C$.

Assume $Q = l_2 \cap l_3$. Prove that $FQ \perp QP$.

(Hint: show that Q is the middle point of RF)

(Remark: we can then obtain all tangent lines and C as their envelope as shown)

<u>Proof</u>: Let $R \in l_1$ s.t. $PR \perp l_1$. Connect FR. Assume $FR \cap l_3 = Q'$.

$$\left.\begin{array}{l} P \in \text{Parabola} \Rightarrow |PR| = |PF|. \quad 4 \\ l_3 \text{ tangent parabola} \Rightarrow \angle RPQ' = \angle FPQ' \quad 4 \end{array}\right\} \overset{SAS}{\Longrightarrow} \triangle RPQ' \overset{\text{congruent}}{\cong} \triangle FPQ'.$$

$\Rightarrow Q'$ is the middle point of FR and $\angle FQ'P = \angle RQ'P = \frac{\pi}{2}$ 4

Just need to prove $Q = Q'$. Since $Q' \in l_3$, it suffices to prove $Q' \in l_2$.

By the definition of l_2, just need to prove $Q'A \parallel l_1$.

Because Q' bisects $\overline{FR}$, $\dfrac{|FQ'|}{|FR|} = \dfrac{1}{2} = \dfrac{|FA|}{|FB|}$. 4

By the Inverse Thales Thm, $Q'A \parallel \ell_1$. The proof is completed. ∎
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Alternate proof:

$$\left.\begin{aligned} |PR| &= |PF| \\ \angle RPQ &= \angle FPQ \end{aligned}\right\} \xrightarrow{SAS} \triangle RPQ \cong \triangle FPQ$$

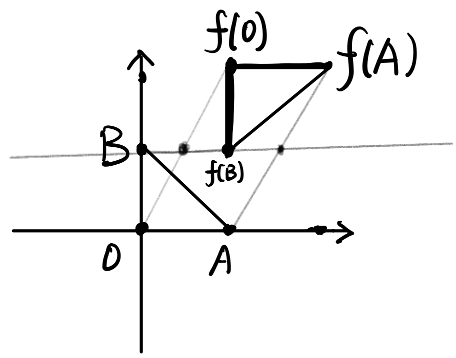
$$\Rightarrow \left.\begin{aligned} |RQ| &= |FQ| \\ |RS| &= |AF| \\ \angle RSQ = \tfrac{\pi}{2} &= \angle FAQ \end{aligned}\right\} \xrightarrow{\text{Pythagorean}} \underset{\sqrt{|RQ|^2-|RS|^2}}{\underset{\shortparallel}{|SQ|}} = \underset{\sqrt{|FQ|^2-|AF|^2}}{\underset{\shortparallel}{|AQ|}} \xrightarrow{SSS} \triangle RQS \cong \triangle FQA$$

$\Rightarrow \angle RQS = \angle FQA \Rightarrow Q$ lies on the line FR and hence $\angle RQP + \angle FQP = \pi$

$$\xrightarrow{\triangle RPQ \cong \triangle FPQ} \angle RQP = \angle FQP = \frac{\pi}{2} \Rightarrow FQ \perp QP \qquad \square$$

3、 Consider the following plane isometry:

$O=(0,0)$
$A=(1,0)$
$B=(0,1)$



$f(0)=(1,2)$
$f(A)=(2,2)$
$f(B)=(1,1)$

(1) Classify f as one of three types (translation, rotation or glide reflection)

(2) Find the translation vector if it is a translation.
or the center and the angle for a rotation
or the axis and the translation vector for a glide reflection.

(1) f changes the orientation $\Rightarrow f$ is a glide reflection.

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(2) The axis contains middle point of $\overline{Of(0)}$:

$$\frac{1}{2}\big((0,0)+(1,2)\big)=\left(\frac{1}{2},1\right)$$

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and middle point of $\overline{Af(A)}$: $\frac{1}{2}\big((1,0)+(2,2)\big)=\left(\frac{3}{2},1\right)$

middle point of $\overline{Bf(B)}$: $\frac{1}{2}\big((0,1)+(1,1)\big)=\left(\frac{1}{2},1\right)$

Axis: $y=1$. direction vector $u=(1,0)$. 6

Reflection: $(0,0) \longmapsto (0,2)$

$(1,0) \longmapsto (1,2)$ 6

$(0,1) \longmapsto (0,1)$

translation vector:

$$f(0)-(0,2)=(1,2)-(0,2)=(1,0)$$

$$f(A)-(1,2)=(2,2)-(1,2)=(1,0)$$ 6

$$f(B)-(0,1)=(1,1)-(0,1)=(1,0).$$

4. (i) Classify the following conic curve:

$$3x^2-10x\cdot y+3y^2+4\sqrt{2}x+4\sqrt{2}y=0$$

(ii) Find its center for an ellipse or hyperbola, or find the vertex for a parabola

(i) Equation: $(x\ \ y)\begin{pmatrix}3 & -5\\ -5 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+4\sqrt{2}(1,1)\begin{pmatrix}x\\ y\end{pmatrix}=0.$ 4

Diagonalize $A=\begin{pmatrix}3 & -5\\ -5 & 3\end{pmatrix}$ by orthogonal matrix:

$$|\lambda I-A|=\begin{vmatrix}\lambda-3 & 5\\ 5 & \lambda-3\end{vmatrix}=\lambda^2-6\lambda-16=(\lambda+2)\cdot(\lambda-8)=0$$

$\Rightarrow \lambda=-2,\ 8$ (with opposite sign). 4

$\Rightarrow$ The curve is a hyperbola. 4

Find axis by calculating eigenvectors:

$\lambda=-2$: $\begin{pmatrix}-5 & 5\\ 5 & -5\end{pmatrix}\to\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}\Rightarrow f_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$ 4

$\Downarrow$

$\lambda=8$: $\begin{pmatrix}5 & 5\\ 5 & 5\end{pmatrix}\to\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\Rightarrow f_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 1\end{pmatrix}$

$\Rightarrow$ $S=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$ $\begin{pmatrix}x\\ y\end{pmatrix}=S\begin{pmatrix}u\\ v\end{pmatrix}$

linear term: $4\sqrt{2}\,(1\ \ 1)\begin{pmatrix} x \\ y \end{pmatrix} = 4\sqrt{2}\,(1\ \ 1)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$$= 4(2,0)\begin{pmatrix} u \\ v \end{pmatrix} = 8u. \qquad 4$$

$\Rightarrow$ Equation in (u,v)-coordinates:

$$-2u^2+8v^2+8u=0 \qquad 4$$

$$\Leftrightarrow \quad u^2-4u-4v^2=0$$

$$\Leftrightarrow \quad \underbrace{u^2-4u+4}_{\shortparallel\ (u-2)^2} -4v^2=4$$

$$(u-2)^2-4v^2=4 \qquad 2$$

$$\Leftrightarrow \quad z_1^2-4z_2^2=4 \quad \text{with} \quad \begin{cases} z_1=u-2 \\ z_2=v \end{cases}$$

Center: $(u,v)=(2,0) \Rightarrow$ center in (x,y) coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} \qquad 4$$