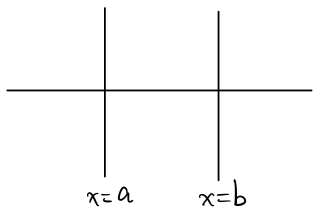
7.1.3



r_1 = reflection across $x=a$

r_2 = reflection across $x=b$

$$r_1 r_2(x,y) = r_1(2b-x, y) = (2a-2b+x, y)$$

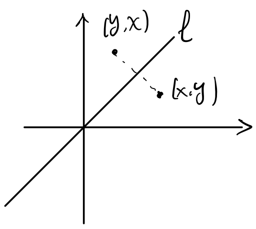
$$r_2 r_1(x,y) = r_2(2a-x, y) = (2b-2a+x, y).$$

So $r_1 r_2 \neq r_2 r_1$ if $a \neq b$.

7.2.2

$$R = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

7.2.3



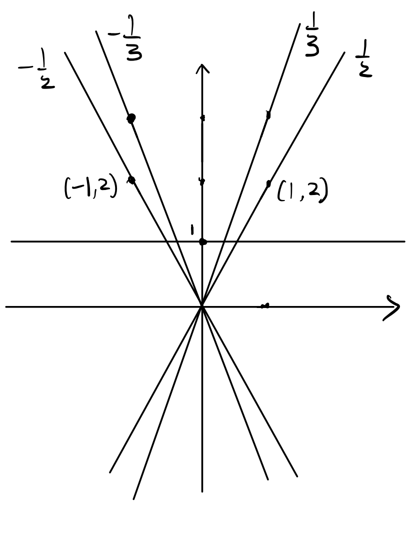
$$r_\ell \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$R\bar{X}R^{-1} = R\bar{X}R^t = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ✓$$

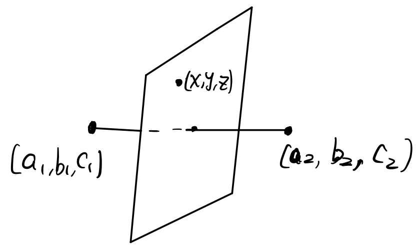
7.3.1. $s \mapsto \frac{1}{s} = \frac{0 \cdot s + 1}{1 \cdot s + 0} \rightsquigarrow M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

7.3.3



$y = k \cdot x$ labeled as $\frac{1}{k}$

7.4.1 :



$$(x-a_1)^2+(y-b_1)^2+(z-c_1)^2=(x-a_2)^2+(y-b_2)^2+(z-c_2)^2$$

$$\Leftrightarrow 2\left((a_2-a_1)x+(b_2-b_1)y+(c_2-c_1)z\right) = a_2^2+b_2^2+c_2^2-a_1^2-b_1^2-c_1^2$$

If $a_1^2+b_1^2+c_1^2=a_2^2+b_2^2+c_2^2=1$, then the plane contains $0\in\mathbb{R}^3$ which becomes

$$(a_2-a_1)x+(b_2-b_1)y+(c_2-c_1)z=0 .$$

7.4.2 : Equidistant set of two points is the intersection of a plane passing through $0\in\mathbb{R}^3$ with the sphere, which is always a great circle.

7.4.3 : $P\neq Q\in S^2$, $A,B,C\in S^2$.

$$\left.\begin{array}{l} dist(P,A)=dist(Q,A) \\ dist(P,B)=dist(Q,B) \\ dist(P,C)=dist(Q,C) \end{array}\right\} \Rightarrow A,B,C \text{ are contained in a great circle.}$$

Equivalently, if A,B,C are not in a "line", then

P,Q have the same distance to $A,B,C \Rightarrow P=Q$.

7.4.4 : $f,g: S^2\to S^2$ both isometry of S^2. $A,B,C\in S^2$ not in a line.

Assume $f(A)=g(A)$ $f(B)=g(B)$ $f(C)=g(C)$.

Then $\forall P\in S^2$, $dist(f(P),f(A))=dist(P,A)=dist(g(P),g(A))$

$dist(f(P),f(B))=dist(P,B)=dist(g(P),g(B))$

$dist(f(P),f(C))=dist(P,C)=dist(g(P),g(C))$

By 7.4.3, $f(P)=g(P)$. So $f=g$.

7.4.5 Let $f: S^2 \to S^2$ be an isometry of S^2.

Choose 3 points $A, B, C \in S^2$ not in a line.

- If $f(A) = A$, then set $g_1 = Id_{S^2}$ (identity transformation). Otherwise, Let g_1 be a reflection that maps A to $f(A)$. Set $f_1 = g_1 \circ f$.

  Then f_1 is an isometry that fixes A.

- If $f_1(B) = B$, then set $g_2 = Id_{S^2}$. Otherwise

  Let g_2 be the reflection across the great circle L_1 that is equidistant set of points B and $f_1(B)$.

  Since $dist(A,B) = dist(f_1(A), f_1(B)) = dist(A, f(B))$, A is contained in L_1

  So g_2 fixes A and $g_2 \circ f_1(B) = B$. Set $f_2 = g_2 \circ f_1$.

  Then f_2 fixes both A and B.

- If $f_2(C) = C$, then set $g_3 = Id_{S^2}$. Otherwise.

  Let g_3 be a reflection across a great circle L_2 that is equidistant set of points C and $f_2(C)$.

  Since $dist(A, C) = dist(f_2(A), f_2(C)) = dist(A, f_2(C))$, $A \in L_2$

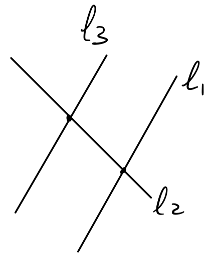
  $dist(B, C) = dist(f_2(B), f_2(C)) = dist(B, f_2(C))$, $B \in L_2$

  So g_3 fixes A, B, and $g_3 \circ f_2(C) = C$. Set $f_3 = g_3 \circ f_2$.

Then f_3 fixes $A, B, C \Rightarrow Id_{S^2} = f_3 \circ g_3 \circ g_2 \circ g_1$

$\Rightarrow f_3 = g_1^{-1} \circ g_2^{-1} \circ g_3^{-1} = g_1 \circ g_2 \circ g_3$ is a product of at most 3 reflections.

7.5.1



$l_1 \parallel l_3$

$$\underbrace{r_{l_1} \circ r_{l_3}}_{\substack{\uparrow \\ \text{translation}}} = \underbrace{(r_{l_1} \circ r_{l_2})}_{\substack{\uparrow \\ \text{rotation}}} \circ \underbrace{(r_{l_2} \circ r_{l_3})}_{\substack{\uparrow \\ \text{rotation}}}$$

7.5.2 R_1 = rotation around P by angle θ_1, $-\pi < \theta_1 \leq \pi$

R_2 = rotation around Q by angle θ_2, $-\pi < \theta_2 \leq \pi$

If $P = Q$, then $R_1 \circ R_2$ is a rotation around P.

Assume $P \neq Q$, then we can decompose:

$$R_1 = r_{l_2} \circ r_{l_1}, \quad R_2 = r_{l_1} \circ r_{l_3}$$

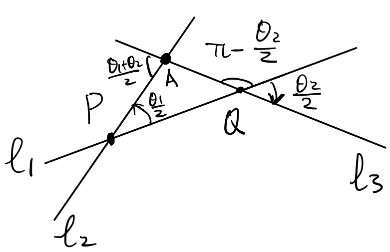
where l_1 is the line containing P and Q.

l_2 is obtained from l_1 by rotating it around P with angle $\frac{\theta_1}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$

l_3 is - - - - - - - - - - - - around Q - - - - - $-\frac{\theta_2}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

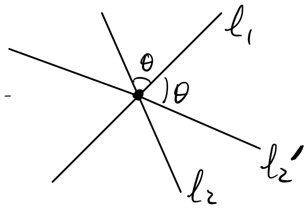
$R_1 \circ R_2 = r_{l_2} \circ r_{l_1} \circ r_{l_1} \circ r_{l_3} = r_{l_2} \circ r_{l_3}$ is a rotation if and only if l_2 is parallel to l_3, if and only if $\frac{\theta_1}{2} = -\frac{\theta_2}{2}$

if and only if $\theta_1 + \theta_2 = 0 \pmod{2\pi}$



So $R_1 \circ R_2$ is a rotation if $\theta_1 + \theta_2 \neq 0 \pmod{2\pi}$, in which case it is a rotation around A with angle $2\left(\frac{\theta_1+\theta_2}{2}\right) = \theta_1 + \theta_2$.

7.5.3. Example:



$$(r_{\ell_2}\circ r_{\ell_1})\circ r_{\ell_1} = r_{\ell_2}\circ(r_{\ell_1}\circ r_{\ell_1}) = r_{\ell_2}$$

✗ 对 $\theta\neq\frac{\pi}{2}$

$$r_{\ell_1}\circ(r_{\ell_2}\circ r_{\ell_1}) = r_{\ell_1}\circ(r_{\ell_1}\circ r_{\ell_2'}) = r_{\ell_2'}$$

$$\parallel$$

$$(r_{\ell_1}\circ r_{\ell_1})\circ r_{\ell_2'}$$