5.7.1 $\quad [p,q,r,s] = \dfrac{r-p}{r-q} \cdot \dfrac{s-q}{s-p} = y$

$$\Longleftrightarrow \quad (r-p)\cdot(s-q) = (r-q)(s-p)\cdot y$$

$$\Longleftrightarrow \quad s\cdot\big((r-q)y-(r-p)\big) = (r-q)\,p\cdot y-(r-p)\cdot q$$

$$\Longleftrightarrow \quad s = \frac{(r-q)p\cdot y-(r-p)q}{(r-q)y-(r-p)} \quad \text{is uniquely determined.}$$

5.7.2 $\quad p=0,\ q=2,\ r=3,\ y=\frac{4}{3}$

$$\Rightarrow s = \frac{(3-2)0\cdot\frac{4}{3}-(3-0)\cdot 2}{(3-2)\cdot\frac{4}{3}-(3-0)} = \frac{-6}{-\frac{5}{3}} = \frac{18}{5}.$$

Verify: $[p,q,r,s] = [0,2,3,\frac{18}{5}] = \dfrac{3-0}{3-2}\cdot\dfrac{\frac{18}{5}-2}{\frac{18}{5}-0} = 3\cdot\dfrac{8}{18} = \dfrac{4}{3}$ ✓

5.8.1 - 5.8.3 Verify directly

5.8.4. $\quad f(y)=\frac{1}{y},\quad g(y)=1-y.\quad f\circ f = Id = g\circ g.$

$$f\circ g(y) = f(1-y) = \frac{1}{1-y}\ .\quad g\circ f(y) = g\left(\tfrac{1}{y}\right) = 1-\frac{1}{y} = \frac{y-1}{y}$$

$$f\circ g\circ f(y) = f\circ g\left(\tfrac{1}{y}\right) = \frac{1}{1-\frac{1}{y}} = \frac{y}{y-1}.$$

$$f\circ g\circ f\circ g(y) = f\circ g\left(\tfrac{1}{1-y}\right) = \frac{1}{1-\frac{1}{1-y}} = \frac{1-y}{-y} = 1-\frac{1}{y} = g\circ f(y).$$

$$g\circ f\circ g(y) = g\circ f(1-y) = \frac{1-y-1}{1-y} = \frac{-y}{y-1} = \frac{y}{y-1} = f\circ g\circ f(y).$$

$$g\circ f\circ g\circ f = f\circ g\circ f\circ f = f\circ g \qquad (f\circ g)^3 = Id = (g\circ f)^3$$

$$\Updownarrow$$

$\Rightarrow$ all compositions:

$$\begin{array}{cccccc} Id, & f, & g, & f\circ g, & g\circ f, & f\circ g\circ f = g\circ f\circ g \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow \\ y & \frac{1}{y} & 1-y & \frac{1}{1-y} & \frac{y-1}{y} & \frac{y}{y-1} \end{array}$$

5.8.5. A permutation can be decomposed as a composition of transposes switching 2 elements.

Any transpose can be decomposed as a composition of transposing nearby elements.

Ex: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)\cdot(23) = (12)(23)(34)(23)(12)\cdot(23)$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1\,4\,3\,2) = (34)(23)(12)$

$$(a_1\, a_2\, a_3 \cdots a_k) = (a_2a_3)(a_3a_4)\cdots(a_{k-1}a_k)(a_1\,a_k)$$

5.8.6. $y = [p,q,r,s] = \frac{r-p}{r-q} \cdot \frac{s-q}{s-p}$

$[q,p,r,s] = \frac{r-q}{r-p} \cdot \frac{s-p}{s-q} = \frac{1}{y}.$

$[p,r,q,s] = \frac{q-p}{q-r} \cdot \frac{s-r}{s-p} = z$

$$y+z = \frac{(r-p)(s-q)-(q-p)(s-r)}{(r-q)(s-p)} = \frac{rs-\cancel{rq}-\cancel{ps}+pq-(qs-\cancel{qr}-\cancel{ps}+pr)}{(r-q)(s-p)}$$

$$= \frac{s(r-q)+p(q-r)}{(r-q)(s-p)} = \frac{(r-q)(s-p)}{(r-q)(s-p)} = 1 \Rightarrow z = 1-y.$$

$[p,q,s,r] = \frac{s-p}{s-q} \cdot \frac{r-q}{r-p} = \frac{1}{y}.$

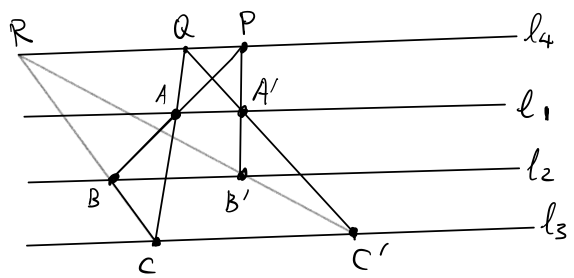
5.8.4

5.8.5 $\Rightarrow$ Any permutation of p,q,r,s produces a cross-ratio that is one of

$$\left\{ y,\ \frac{1}{y},\ 1-y,\ 1-\frac{1}{y},\ \frac{1}{1-y},\ \frac{y}{y-1} \right\}$$

$$\begin{array}{cccccc} \| & \| & \| & \| & \| & \| \\ Id & f & g & g\circ f & f\circ g & f\circ g\circ f \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ [p,q,r,s] & [q,p,r,s] & [p,r,q,s] & [q,r,p,s] & [r,p,q,s] & [r,q,p,s] \end{array}$$

6.2.1:

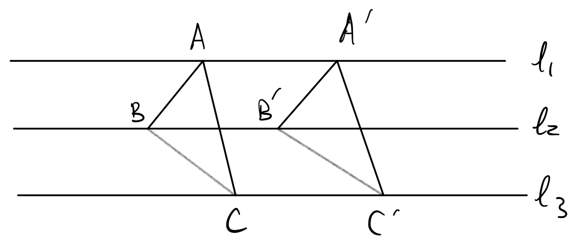


$AB \cap A'B' = P$

$AC \cap A'C' = Q$

$PQ \parallel l_1 \parallel l_2 \parallel l_3 \Leftrightarrow PQ\,[l]$ on a line

$\Rightarrow BC \cap B'C' = R \in$ line PQ



$\Leftrightarrow AB \cap A'B' = [AB] \in$ horizon

$AB \parallel A'B'$

$\Leftrightarrow AC \cap A'C' = [AC] \in$ horizon.

$AC \parallel AC'$

$l_1 \parallel l_2 \parallel l_3 \quad \Rightarrow BC \parallel B'C'$

$\Updownarrow$

$[l_1] = [l_2] = [l_3] \in$ horizon

6.2.2

6.2.4