5.3.1: $\begin{cases} a\cdot 0 + b\cdot 0 + c\cdot 1 = 0 \Rightarrow c = 0 \\ a\cdot 1 + b\cdot 1 + c\cdot 1 = 0 \end{cases} \Rightarrow \begin{cases} a = -b \\ c = 0. \end{cases} \quad \forall b \in \mathbb{R}.$

$$\Rightarrow \underbrace{-b\cdot x + b\cdot y}_{= -b(x-y) \ \forall b \neq 0} = 0 \iff x - y = 0 \text{ is the plane.}$$

$(1,0,0)$, $(0,1,0)$ does not satisfy $x-y=0$ $\Rightarrow$ they are not on the plane

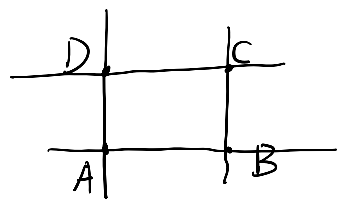
5.3.2. Take 4 "points": $\underset{A}{(1,0,0)}$, $\underset{B}{(0,1,0)}$, $\underset{C}{(0,0,1)}$, $\underset{D}{(1,1,1)}$.

The line: $\overline{AB}$: $z=0$

$\overline{BC}$: $x=0$

$\overline{CD}$: $x-y=0$

$\overline{DA}$: $y-z=0$



No three of which have a common point.

5.4.1: $\begin{cases} a\cdot 1 + b\cdot 2 + c\cdot 3 = 0 \\ a\cdot 1 + b\cdot 1 + c\cdot 1 = 0 \end{cases} \iff \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \underset{\text{(rref)}}{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}} \iff \begin{cases} a = c \\ b = -2c \end{cases} \quad \forall c \in \mathbb{R}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ -2c \\ c \end{pmatrix} = c\cdot\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \text{The "line"} \ \boxed{x - 2y + z = 0}$$

= plane passing through $0 \in \mathbb{R}^3$

5.4.2 – 5.4.3:

$$\begin{cases} x+2y+3z=0 \\ x+y+z=0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \forall z\in\mathbb{R}$$

$\Rightarrow$ intersection "point" is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ (homogeneous coordinate)

$\parallel$

line passing through O: $\mathbb{R}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

5.6.1: $y=\dfrac{ax+b}{cx+d}$ $\qquad ad-bc\neq 0$

$$\Rightarrow \quad y\cdot(cx+d)=ax+b \Leftrightarrow (c\cdot y-a)x=b-dy$$

$$\Rightarrow \quad x=\frac{b-dy}{cy-a} \quad \text{if } cy-a\neq 0.$$

If $c\cdot y-a=0$, then $c\neq 0$ (otherwise $a=c=0 \Rightarrow ad-bc=0$)

and $b-d\cdot y\neq 0$ (otherwise $ad-bc=-d\cdot(cy-a)-c\cdot(b-dy)=0$)

$$\Rightarrow \quad y=\frac{a}{c} \quad \text{and} \quad x=\frac{b-dy}{0}=\frac{b-d\cdot\frac{a}{c}}{0}=\infty.$$

5.6.2 – 5.6.3.

$$f_1(f_2(x))=\frac{a_1\cdot f_2(x)+b_1}{c_1\cdot f_2(x)+d_1}=\frac{a_1\cdot\frac{a_2x+b_2}{c_2x+d_2}+b_1}{c_1\cdot\frac{a_2x+b_2}{c_2x+d_2}+b_1}$$

$$=\frac{(a_1a_2+c_2b_1)x+(a_1b_2+b_1d_2)}{(c_1a_2+c_2b_1)x+(c_1b_2+d_2b_1)}=\frac{Ax+B}{Cx+D}.$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{pmatrix}$$

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \checkmark$$

5.6.4 : $AD-BC = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \det\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$$= (a_1d_1 - b_1c_1)\cdot(a_2d_2 - b_2c_2) \neq 0 \, ,$$