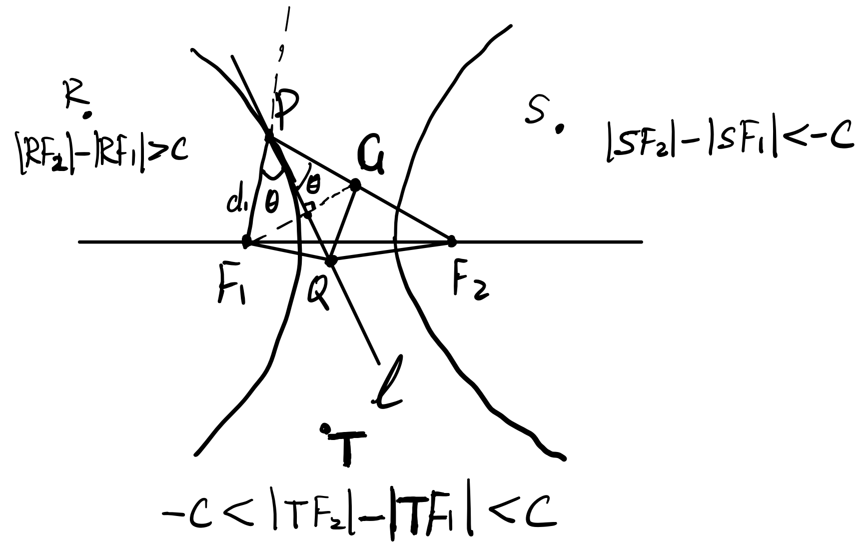
1.



The tangent line ℓ at P is the bisection line of the angle $\angle F_1PF_2$

Proof:

- Set $|PF_1|=d_1$, $|PF_2|=d_2$, $|PG|=|PF_1|=d_1$

  Then $|GF_2|=|PF_2|-|PG|=d_2-d_1=\text{constant}=c$.

- For any Q on the bisection line ℓ, by SAS,

  $$\Delta F_1PQ \cong \Delta GPQ \Rightarrow |F_1Q|=|GQ|$$

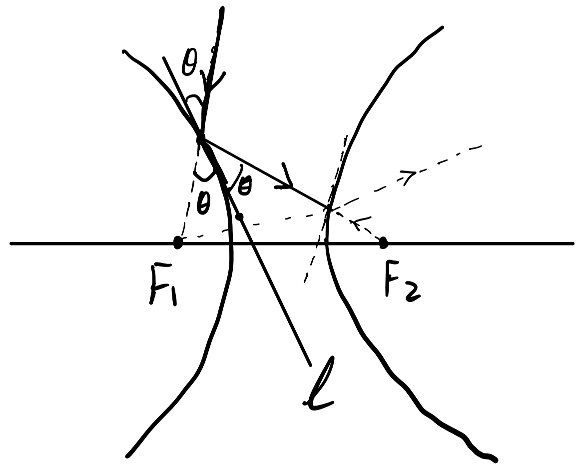
- In the triangle ΔF_1QF_2, by triangle inequality

$$|QF_2|-|QF_1|$$
$$\parallel$$
$$-|F_2G|<|QF_2|-|QG|<|F_2G|$$

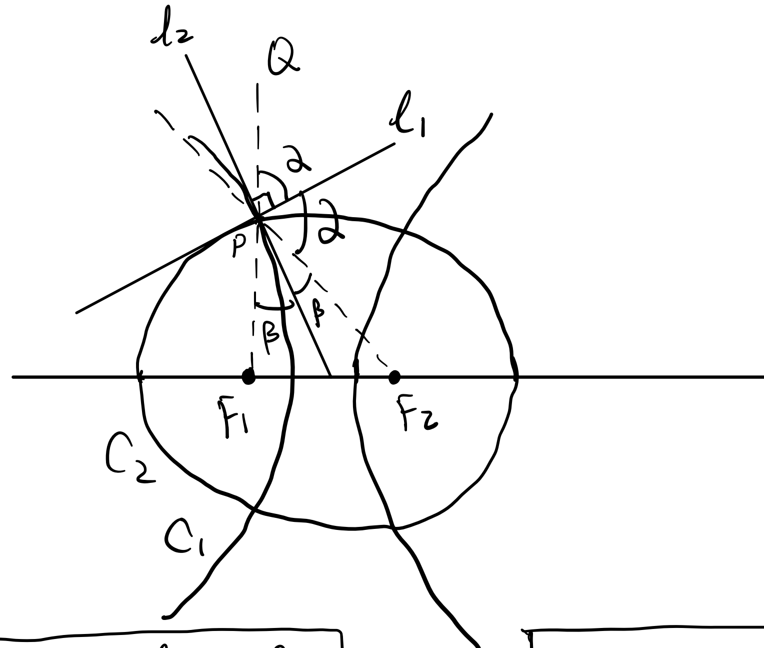
$\Rightarrow$ Q lies between the two branches

$\Rightarrow$ ℓ is tangent to the hyperbola.

This also explain the 2nd statement.



2.



$\boxed{\text{The tangents } l_1 \perp l_2} \Longleftrightarrow \boxed{C_1 \perp C_2 \text{ at intersection points.}}$

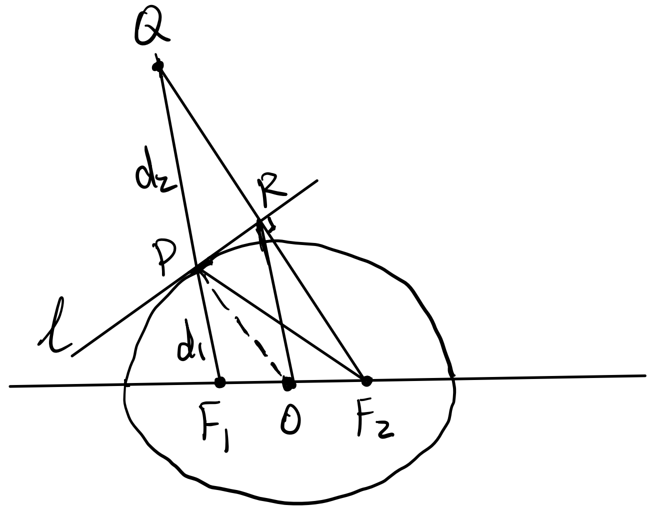
Proof: l_2 bisects $\angle F_1PF_2 = 2\beta$

l_1 bisects $\angle F_2PQ = 2\alpha$

$2\alpha + 2\beta = 2\pi$

$\Rightarrow \alpha + \beta = \frac{\pi}{2}$

3.



- Extend F_1P by $d_2 = |PF_2|$ to get Q.
  Then $|F_1Q| = d_1 + d_2 = 2a$
- Because the tangent line ℓ bisects $\angle F_2PQ$,
  $R = \ell \cap \overline{F_2Q}$ is the middle point of $\overline{F_2Q}$.
- Because $\dfrac{|F_2R|}{|F_2Q|} = \dfrac{|F_2O|}{|F_1O|}$, by inverse Thales thm,

$OR \parallel F_1Q$ and $\triangle F_2OR$ is similar to $\triangle F_2F_1Q$.

So $\dfrac{|OR|}{|F_1Q|} = \dfrac{|F_2O|}{|F_1O|} = \dfrac{1}{2} \Rightarrow |OR| = \dfrac{1}{2} \cdot |F_1Q| = a$

constant.

$\Rightarrow$ R lies on the circle with center O and radius a.

4. (i) $11x^2+6xy+3y^2-12x-12y-12=0$. $(*)$

step 1: diagonalize $A=\begin{pmatrix}11 & 3\\ 3 & 3\end{pmatrix}$ by orthogonal matrix S.

$$|\lambda I-A|=\begin{vmatrix}\lambda-11 & -3\\ -3 & \lambda-3\end{vmatrix}=\lambda^2-14\lambda+24=(\lambda-2)(\lambda-12).$$

$\Rightarrow \lambda=2, 12.$

$\lambda=2$: $\begin{pmatrix}-9 & -3\\ -3 & -1\end{pmatrix}\to\begin{pmatrix}3 & 1\\ 0 & 0\end{pmatrix}\Rightarrow \tilde{f}_1=\begin{pmatrix}1\\ -3\end{pmatrix}\Rightarrow f_1=\frac{1}{\sqrt{10}}\begin{pmatrix}1\\ -3\end{pmatrix}$

$\Rightarrow \lambda=12$: $f_2=\frac{1}{\sqrt{10}}\cdot\begin{pmatrix}3\\ 1\end{pmatrix}$.

$\Rightarrow S=(f_1\ f_2)=\frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3\\ -3 & 1\end{pmatrix}$. $\begin{pmatrix}x\\ y\end{pmatrix}=S\begin{pmatrix}u\\ v\end{pmatrix}$.

step 2: Calculate the transformation of the linear terms:

$$-12x-12y=(-12\ \ -12)\begin{pmatrix}x\\ y\end{pmatrix}=(-12\ -12)\cdot\frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3\\ -3 & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

$$=\frac{1}{\sqrt{10}}(-12+36\ \ -36-12)\begin{pmatrix}u\\ v\end{pmatrix}=\frac{1}{\sqrt{10}}(24, -48)\begin{pmatrix}u\\ v\end{pmatrix}$$

$$=\frac{1}{\sqrt{10}}(24u-48v).$$

$(\ast) \longrightarrow 2u^2 + 12v^2 + \frac{24}{\sqrt{10}}u - \frac{48}{\sqrt{10}}v - 12 = 0$

$$\Leftrightarrow 2\left(u^2 + \frac{12}{\sqrt{10}}u + \left(\frac{6}{\sqrt{10}}\right)^2\right) + 12\cdot\left(v^2 - \frac{4}{\sqrt{10}}v + \left(\frac{2}{\sqrt{10}}\right)^2\right)$$

$$= 12 + 2\cdot\frac{36}{10} + 12\cdot\frac{4}{10}$$

$$= 12 + \frac{72+48}{10} = 12 + 12 = 24$$

$$\Leftrightarrow 2\left(u + \frac{6}{\sqrt{10}}\right)^2 + 12\left(v - \frac{2}{\sqrt{10}}\right)^2 = 24$$

$\Rightarrow$ set $\begin{cases} z_1 = u + \frac{6}{\sqrt{10}} \\ z_2 = v - \frac{2}{\sqrt{10}} \end{cases}$. $\frac{z_1^2}{12} + \frac{z_2^2}{2} = 1$. is an ellipse.

step3

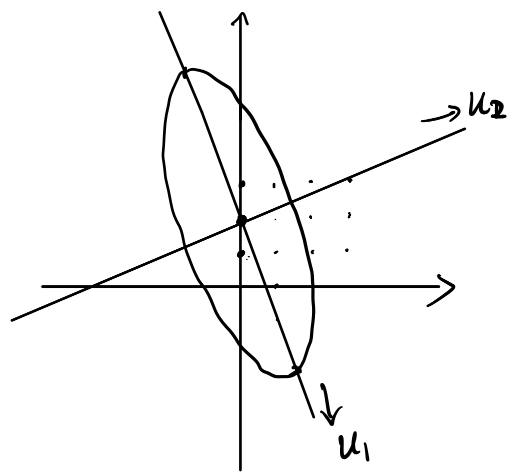
- center of ellipse in (u,v)-coordinates: $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} -\frac{6}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{pmatrix}$

$\Rightarrow$ in (x,y) coordinates,

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = S\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} -\frac{6}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -6+6 \\ 18+2 \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

direction of axes: $u_1 = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\-3\end{pmatrix}$, $u_2 = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix}$



(ii) $x^2-2xy+y^2-10x-6y+25=0$

- $A=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$ $\quad |\lambda I-A| = \begin{vmatrix}\lambda-1&1\\1&\lambda-1\end{vmatrix} = \lambda^2-2\lambda=\lambda(\lambda-2)=0$

$\Rightarrow \lambda=0,2.$

$\lambda=0$: $\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\to\begin{pmatrix}1&-1\\0&0\end{pmatrix}\Longrightarrow f_1=\frac{1}{\sqrt{1^2+1^2}}\begin{pmatrix}1\\1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$

$\Longrightarrow \lambda=2$: $f_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}.$ $\quad S=(f_1\ f_2)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}.$

- $(-10\ \ -6)\begin{pmatrix}x\\y\end{pmatrix}=(-10\ \ -6)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}=\frac{1}{\sqrt{2}}(-10-6\quad 10-6)=\frac{1}{\sqrt{2}}(-16,4).$

$\rightsquigarrow\ 2v^2-\frac{16}{\sqrt{2}}u+\frac{4}{\sqrt{2}}v+25=0$

$$\Leftrightarrow \quad 2\left(v^2+\frac{2}{\sqrt2}v+\left(\frac{1}{\sqrt2}\right)^2\right)=\frac{16u}{\sqrt2}-25+1=\frac{16}{\sqrt2}\cdot\left(u-\frac{24}{16}\sqrt2\right)$$

$$\Leftrightarrow \quad 2\left(v+\frac{1}{\sqrt2}\right)^2=\frac{16}{\sqrt2}\cdot\left(u-\frac{3}{2}\sqrt2\right).$$

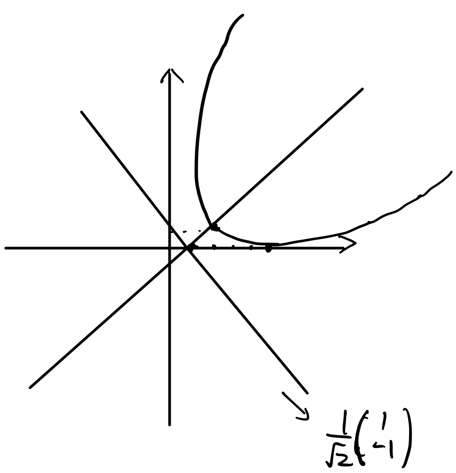
$$\Rightarrow \quad \frac{\sqrt2}{8}z_2^2=z_1 \quad \text{with } z_1=u-\frac{3}{2}\sqrt2,\ z_2=v+\frac{1}{\sqrt2}$$

This is a parabola with vertex $(u_0,v_0)=\left(\frac{3}{2}\sqrt2,-\frac{\sqrt2}{2}\right)$.

$\rightsquigarrow$ in (x,y) coordinates, the vertex is given by:

$$\begin{pmatrix}x_0\\y_0\end{pmatrix}=\frac{1}{\sqrt2}\cdot\begin{pmatrix}1&-1\\1&1\end{pmatrix}\cdot\begin{pmatrix}\frac{3}{2}\sqrt2\\-\frac{\sqrt2}{2}\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}.$$

The direction of the axis is $\frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$



5.2.1-5.2.2:   Some steps: