1. Classify the following plane isometry. Determine the translation vector (for translation), center (for rotation), axis and translation vector (for glide reflection).

(1) $$\begin{cases} x' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y + 3 \\ y' = \frac{\sqrt{3}}{2}x + \frac{1}{2}y - 1 \end{cases}$$

(2) $$\begin{cases} x' = \frac{12}{13}x + \frac{5}{13}y - 1 \\ y' = \frac{5}{13}x - \frac{12}{13}y + 5. \end{cases}$$

(3) $$\begin{cases} x' = \frac{4}{5}x - \frac{3}{5}y + 2 \\ y' = -\frac{3}{5}x - \frac{4}{5}y + 2 \end{cases}$$

(1) $$X' = \begin{pmatrix} x' \\ y' \end{pmatrix} = A \cdot X + b \quad \text{with } A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$\det(A) = \frac{1}{4} + \frac{3}{4} = 1 \Rightarrow$ rotation. $\quad b = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

angle θ satisfies $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3} = 60°$.

center X_0 satisfies $A \cdot X_0 + b = X_0$

$$\Rightarrow X_0 = (I-A)^{-1} b = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3+\sqrt{3}}{2} \\ \frac{3\sqrt{3}-1}{2} \end{pmatrix}.$$

(2) $$X' = AX + b \quad \text{with } A = \begin{pmatrix} \frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & -\frac{12}{13} \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$\det A = -\frac{12^2+5^2}{13^2} = -1 \Rightarrow$ glide reflection.

find eigenvector associated to 1:

$1 \cdot I - A = \begin{pmatrix} \frac{1}{13} & -\frac{5}{13} \\ -\frac{5}{13} & \frac{25}{13} \end{pmatrix} \to \begin{pmatrix} 1 & -5 \\ 0 & 0 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ represents the direction of the axis ℓ

A point on ℓ: $\frac{0+f(0)}{2} = \frac{b}{2} = \frac{1}{2}\begin{pmatrix} -1 \\ 5 \end{pmatrix} \Rightarrow$ the axis is given by:

$$\ell: \begin{cases} x = -\frac{1}{2} + 5t \\ y = \frac{5}{2} + t \end{cases} \Leftrightarrow x + \frac{1}{2} = 5\left(y - \frac{5}{2}\right) \Leftrightarrow x - 5y + 13 = 0$$

translation vector: Because $b\cdot u = (-1\ \ 5)\begin{pmatrix}5\\1\end{pmatrix} = -5+5=0$

$$v = \mathrm{Proj}_{\frac{u}{|u|}} b = \frac{b\cdot u}{|u|^2}u = 0 \Rightarrow \text{just a reflection.}$$

(3) $\begin{cases} x' = \frac{4}{5}x - \frac{3}{5}y + 2 \\ y' = -\frac{3}{5}x - \frac{4}{5}y + 2 \end{cases} \Leftrightarrow X' = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{pmatrix} X + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\det A = -\frac{16+9}{25} = -1 \Rightarrow$ glide reflection.

$I - A = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{9}{5} \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \Rightarrow u = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ represents the direction of axis l.

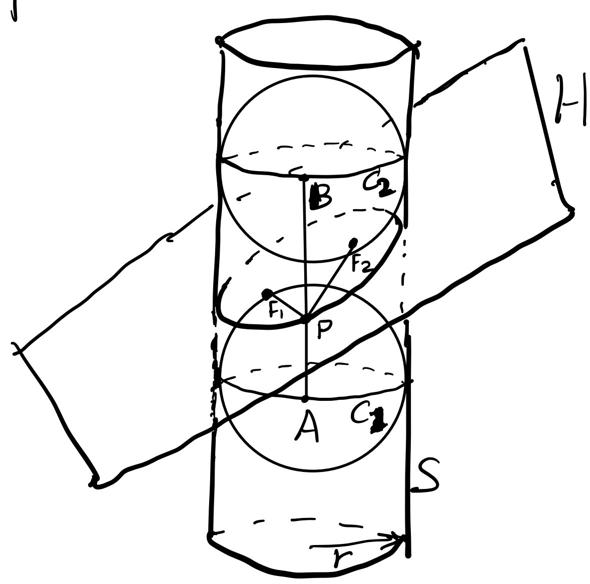
$\frac{b}{2} = \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow l: \begin{cases} x = 1 - 3t \\ y = 1 + t \end{cases} \Leftrightarrow x - 1 + 3\cdot(y-1) = 0 \Leftrightarrow x + 3y - 4 = 0.$

translation vector: $b\cdot u = (1\ \ 1)\begin{pmatrix} -3 \\ 1 \end{pmatrix} = -2$

$$v = \frac{b\cdot u}{|u|^2}u = \frac{-2}{3^2+1}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = -\frac{2}{10}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ -\frac{1}{5} \end{pmatrix}$$

2. Show that a plane section of a circular cylinder is an ellipse.



Put two spheres of radius r to touch the plane H. One at F_2 above H and the other at F_1 below H

For any point $P \in H \cap S$, $|PF_2| = |PB|$, $|PF_1| = |PA|$

$\Rightarrow |PF_1| + |PF_2| = |PA| + |PB| = |AB| =$ distance from C_1 to C_2 which is constant. So we get an ellipse.

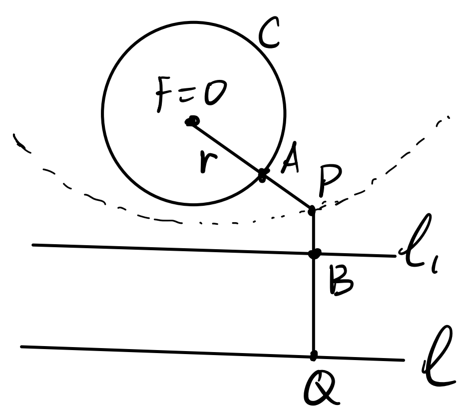
(Here C_1 and C_2 are intersection circles of the cylinder with the spheres.)

Note that if H is perpendicular to the cylinder, then $F_1 = F_2$ which gives us just a circle.

3. Show that the locus of points P that have the same distance from a given circle and a given line are parabola.
(use the definition of parabola, find the focus / directrix)

There are Three cases:

- Easiest case: l_1 is disjoint to the circle C

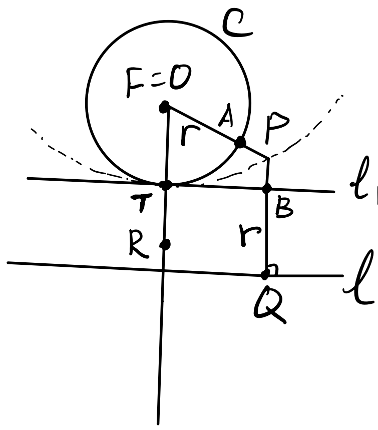


Draw a line l that is parallel to l_1 and has distance r (radius of the circle) below l_1. Then

$$|PA|=|PB|,\ |BQ|=r \Rightarrow |PO|=|PQ|$$

$\Rightarrow$ parabola with focus $F=O$ and directrix l

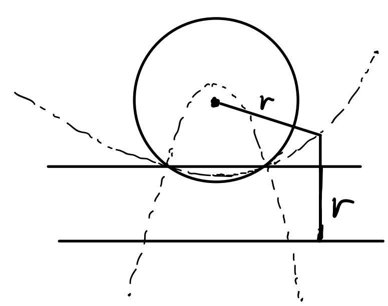
- Degenerate case: when the line ℓ_1 is tangent to the circle C at T



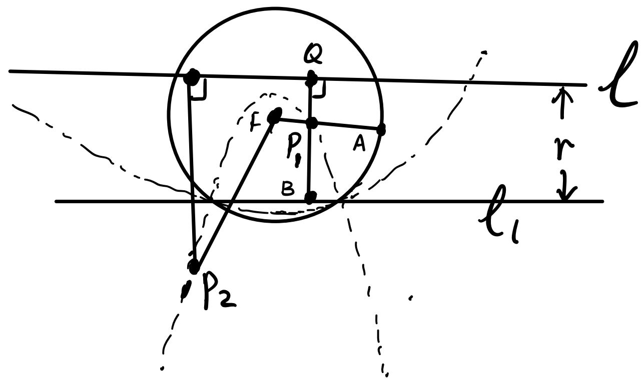
The locus of equidistant points to C and ℓ_1 $=$ a parabola + the ray $\mathbb{R}_+ \cdot \overrightarrow{OT}$

$\text{dist}(R, \ell_1) = |RT| = \text{dist}(R, \text{circle})$ for any R on the ray.

- when the line ℓ_1 intersects the circle, there are two parabolas. The first one is similar to above.



The second one has the directrix l above l_1:



A symmetric position:

when the line passes through O: