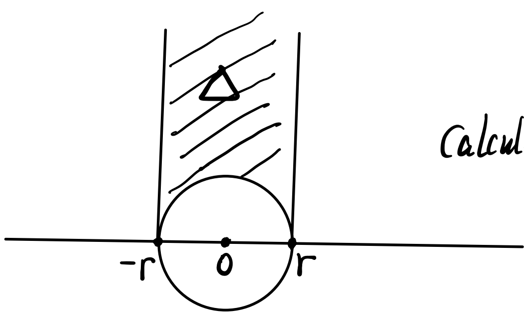
8.5.1 — 8.5.5

1. Use the existence / Uniqueness of the equidistant line for any two points (and the arguments similar to the case of $\mathbb{R}^2$) to complete the proof of 3 reflection theorem for the non-Euclidean geometry (the upper half plane).

2. Use calculus to verify that the area of the limit triangle

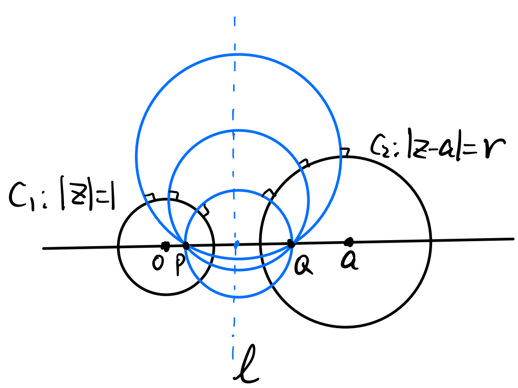
$$\text{Area}(\Delta) = \pi :$$




Calculate double the integral: $\int_{\Delta} \frac{dx\,dy}{y^2}$

Does this integral depend on the radius r?

3. Consider 2 circles (non-Euclid lines) $C_1, C_2$ that do not intersect:




Prove that the circles that are orthogonal to both $C_1$ and $C_2$ have centers lying on a line $\ell$ and all pass through 2 points P and Q.

(Use equations to find equation for $\ell$ and the coordinates for P, Q)

8.5.1 - 8.5.5. See Note 14a.

1. • <u>Lemma</u>: Given 3 points $A, B, C$ that are not on the same Euclidean line. Any point $P$ is uniquely determined by its distances to $A, B, C$.

This depends on <u>the fact</u> that: Given any two points $A$ and $A'$, there is a unique non-Euclidean line that consists of all points with equal distances to $A$ and $A'$.

<u>Another fact</u>: $A$ can be reflected to $A'$ by this equidistant line.

• Let $f: U \to U$ be an isometry. Process to decompose $f$ as at most 3 reflections

Pick 3 points $A, B, C$ not on a line. Denote $f(A) = A'$, $f(B) = B'$, $f(C) = C'$

step 1: If $A = A'$, set $r_1 = Id_U$. If $A \neq A'$, set $r_1 = r_{\ell_1}$ which is the reflection w.r.t. the equi-distant line $\ell_1$ for $\{A, A'\}$.

Then the composition $r_1 \circ f = f_2$ is an isometry satisfying $f_2(A) = A$.

<u>step 2</u>: Denote $B_2 = f_2(B)$, $C_2 = f_2(C)$.

If $B_2 = B$, set $r_2 = Id_U$. If $B \neq B_2$, set $r_2 = r_{\ell_2}$ which is the reflection w.r.t. the equidistant line $\ell_2$ for $\{B, B_2\}$.

Then • $\text{dist}(A, B) = \text{dist}(f_2(A), f_2(B)) = \text{dist}(A, B_2) \Rightarrow A \in \ell_2$.

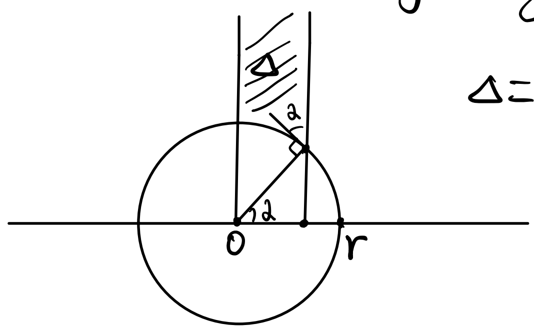
• the composition $r_2 \circ f_2 = f_3$ is an isometry that satisfies:

$f_3(A) = A$, $f_3(B) = B$.

<u>Step 3</u>: Denote $C_3 = f_3(C)$. If $C_3 = C$, set $r_3 = Id_U$, If $C_3 \neq C$, set $r_3 = r_{\ell_3}$ the reflection w.r.t. the equidistant line $\ell_3$ for $\{C, C_3\}$. Then $r_3 \circ f_3 = r_3 \circ r_2 \circ r_1 \circ f$ fixes $A, B, C \Rightarrow r_3 \circ r_2 \circ r_1 \circ f = Id_U$

$\Rightarrow f = r_1 \circ r_2 \circ r_3$ is a composition of at most 3 reflections

2. We calculate more generally:

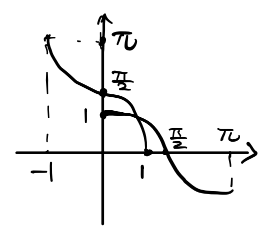



$$\Delta = \Delta_\alpha : \quad 0 \le x \le r\cdot\cos\alpha$$
$$\sqrt{r^2-x^2} \le y < +\infty .$$

$$\Rightarrow \text{Area}(\Delta) = \iint_\Delta \frac{dx\,dy}{y^2} = \int_0^{r\cos\alpha} dx \cdot \int_{\sqrt{r^2-x^2}}^{+\infty} \frac{dy}{y^2} .$$

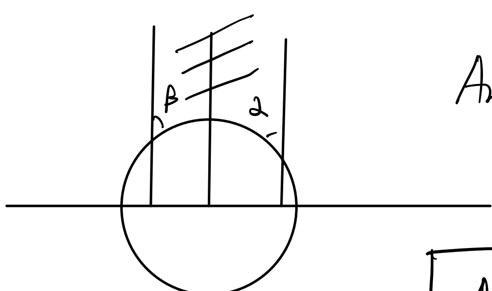
$$= \int_0^{r\cos\alpha} dx \cdot \left(-\frac{1}{y}\right)\Bigg|_{\sqrt{r^2-x^2}}^{+\infty} = \int_0^{r\cos\alpha} \frac{dx}{\sqrt{r^2-x^2}} = \int_0^{r\cos\alpha} \frac{d\left(\frac{x}{r}\right)}{\sqrt{1-\left(\frac{x}{r}\right)^2}}$$

$$\overset{t=\frac{x}{r}}{=\!=} \int_0^{\cos\alpha} \frac{dt}{\sqrt{1-t^2}} = -\cos^{-1}x \Big|_0^{\cos\alpha}$$



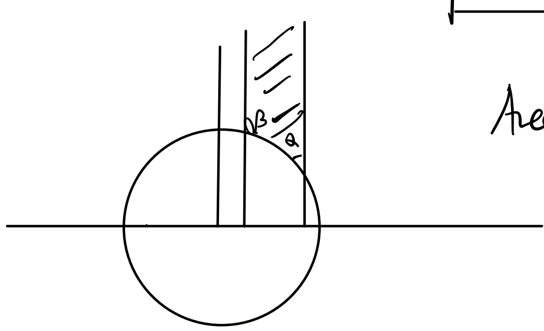

$$= -\left(\alpha - \frac{\pi}{2}\right) = \frac{\pi}{2} - \alpha = \pi - \left(\frac{\pi}{2} + \alpha\right).$$

(Does not depend on $r$ )



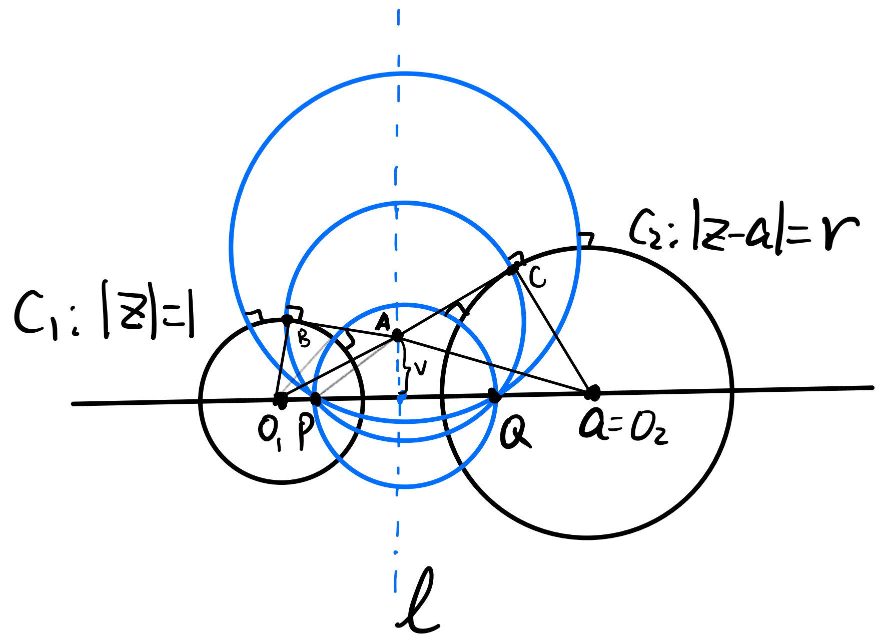

$$\text{Area}(\Delta_{\alpha,\beta}) = \left(\frac{\pi}{2} - \alpha\right) + \left(\frac{\pi}{2} - \beta\right)$$
$$= \pi - (\alpha + \beta)$$

$$\boxed{\text{Area}(\Delta_{0,0}) = \pi - (0+0) = \pi}$$




$$\text{Area}(\Delta_{\alpha,\beta}) = \left(\frac{\pi}{2} - \alpha\right) - \left(\frac{\pi}{2} - (\pi - \beta)\right)$$
$$= \pi - (\alpha + \beta).$$

3.




$$|AB|^2 = |AC|^2 \iff \underbrace{|AO_1|^2 - |O_1B|^2}_{=} = \underbrace{|AO_2|^2 - |O_2C|^2}_{=}$$

Set $A = w$ :  $$|w|^2 - 1 = |w-a|^2 - r^2$$

$$\iff |w|^2 - 1 = |w|^2 - aw - a\bar{w} + a^2 - r^2$$

$$\iff a\underbrace{(w+\bar{w})}_{=\,2a\cdot\mathrm{Re}\,w} = a^2+1-r^2 \iff \mathrm{Re}\,w = \frac{a^2+1-r^2}{2a}.$$

$\Rightarrow$ A lies on the vertical line : $x = \dfrac{a^2+1-r^2}{2a} = x_0$.

The circle centered at A :

$$\underbrace{|z-w|^2}_{=\,|z|^2 - \bar{w}\cdot z - w\bar{z} + |w|^2} = |w|^2 - 1 \qquad w = x_0 + i\cdot v$$

$$\Leftrightarrow \quad |z|^2 - \bar{w}\cdot z - w\cdot\bar{z} + 1 = 0 \qquad z = x + iy.$$

intersection with $x$-axis: $\quad y = 0$

$$x^2 - (\bar{w} + w)\,x + 1 = 0 \quad \Leftrightarrow \quad x^2 - 2x_0\cdot x + 1 = 0$$

$$\Rightarrow \quad x = \frac{x_0 \pm \sqrt{x_0^2 + 1}}{2}$$

coordinates of $P$, $Q$ that does not depend on the height $v$ of the centers.