8.7.1 reflection in the y-axis: $f(z) = -\bar{z}$

in the unit circle: $g(z) = \frac{1}{\bar{z}}$

$$\Rightarrow \quad f\circ g(z) = f\left(\frac{1}{\bar{z}}\right) = -\overline{\left(\frac{1}{\bar{z}}\right)} = -\frac{1}{z}$$

8.7.2 $-\frac{1}{z} = w \Leftrightarrow z = -\frac{1}{w}$

$$|z-ti| = \sqrt{t^2-1} \Leftrightarrow (z-ti)\cdot(\bar{z}+ti) = t^2-1 \Leftrightarrow |z|^2+tiz-ti\bar{z}+1=0$$

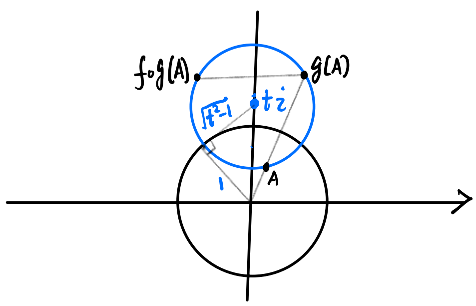
$$(z-ti)\cdot(\bar{z}+ti) = |z|^2+tiz-ti\cdot\bar{z}+t^2$$

$$|z|^2+tiz-ti\bar{z}+1=0 \Leftrightarrow C \Leftrightarrow |w|^2+tiw-ti\bar{w}+1=0$$

$$\Rightarrow \frac{1}{|w|^2} + ti\cdot\left(-\frac{1}{w}\right) - ti\cdot\left(-\frac{1}{\bar{w}}\right) + 1 = 0 \Leftrightarrow |w|^2+tiw-ti\bar{w}+1=0$$

$$\frac{1}{|w|^2} + ti\cdot\left(-\frac{1}{w}\right) - ti\cdot\left(-\frac{1}{\bar{w}}\right) + 1 = \frac{1}{|w|^2}\cdot\left(1-ti\bar{w}+ti\cdot w+|w|^2\right)$$

⇒ same equation as that satisfied by z ⇒ C is mapped to C.



8.7.3 $f(z)=z+1$, $g(z)=-\frac{1}{z}$ $\quad g^{-1}(z)=-\frac{1}{z}$

$$g\circ f\circ g^{-1}(z) = g\circ f\left(-\frac{1}{z}\right)=g\Big(\underbrace{-\frac{1}{z}+1}_{=\frac{z-1}{z}}\Big) = -\frac{1}{\frac{z-1}{z}} = -\frac{z}{z-1}.$$

8.7.4: g^{-1}: rotation around i by angle π: $0 \to \infty$

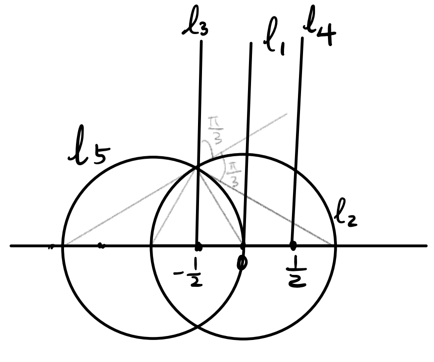
f: limit rotation around ∞

g: rotation around i by angle π: $\infty \to 0$

$$0 \xmapsto{g^{-1}} \infty \xrightarrow{f} \infty \xrightarrow{g} 0$$

$g\circ f\circ g^{-1}$: limit rotation around 0

Some discussion / calculations:



$g = r_{\ell_1}\circ r_{\ell_2} = r_{\ell_2}\circ r_{\ell_1}$

$g^{-1} = r_{\ell_2}\circ r_{\ell_1} = g$

$f = r_{\ell_1}\circ r_{\ell_3} = r_{\ell_4}\circ r_{\ell_1}$

$$g \circ f \circ g^{-1} = r_{\ell_2} \circ r_{\ell_1} \circ r_{\ell_1} \circ r_{\ell_3} \circ r_{\ell_2} \circ r_{\ell_1}$$

$$= r_{\ell_2} \circ r_{\ell_3} \circ r_{\ell_2} \circ r_{\ell_1}$$

$$= r_{\ell_2} \circ r_{\ell_2} \circ r_{\ell_5} \circ r_{\ell_1}$$

$$= r_{\ell_5} \circ r_{\ell_1}$$ limit rotation

$r_{\ell_1}(z) = -\bar{z}$ $\qquad h(z) = z+1$

$$r_{\ell_5}(z) = h^{-1} \circ r_{\ell_2} \circ h(z) = h^{-1} \circ r_{\ell_2}(z+1) = h^{-1}\left(\frac{1}{\bar{z}+1}\right)$$

$$= \frac{1}{\bar{z}+1} - 1 = \frac{-\bar{z}}{\bar{z}+1}$$

$$r_{\ell_5} \circ r_{\ell_1}(z) = r_{\ell_5}(-\bar{z}) = \frac{-\overline{-\bar{z}}}{\overline{-\bar{z}}+1} = \frac{z}{1-z}.$$

8.8.1 $\quad f(x) = \dfrac{ax+b}{cx+d}, \quad f(p) = q, \ f(q) = p$

$$\left.\begin{aligned} \frac{ap+b}{cp+d} = q &\Leftrightarrow ap+b = cpq+dq \\ \frac{aq+b}{cq+d} = p &\Leftrightarrow aq+b = cpq+dp \end{aligned}\right\} \Rightarrow$$

$$a(p-q) = d(q-p) \xRightarrow{p\neq q} a = -d \Leftrightarrow d = -a$$

$$b = cpq + dq - ap = cpq - a(p+q).$$

$$\Rightarrow f(x) = \frac{ax + cpq - a(p+q)}{cx - a} = \frac{a(x-p-q)+cpq}{cx-a}$$

讨 $c \neq 0$ $\quad = \dfrac{\frac{a}{c}(x-p-q)+pq}{x-\frac{a}{c}} \overset{k=\frac{a}{c}}{=} \dfrac{k(x-p-q)+pq}{x-k}$

8.8.2 $\quad x = \dfrac{k(x-p-q)+pq}{x-k} \Leftrightarrow x^2 - kx = k(x-p-q)+pq$

$$\Leftrightarrow x^2 - 2kx + k(p+q) - pq = 0$$

$$\Rightarrow x = \frac{2k \pm \sqrt{(2k)^2 - 4k(p+q) + 4pq}}{2} = k \pm \sqrt{(k-p)(k-q)}.$$

8.8.3 $ad-bc = a\cdot(-a) - (cpq - a(p+q))c$

$= -a^2 - c^2pq + ac(p+q)$

$= -c^2\left(\frac{a^2}{c^2} - \frac{a}{c}(p+q) + pq\right) \overset{k=\frac{a}{c}}{=} -c^2(k-p)(k-q)$

if $f(z)$ preserves the upper half plane, $(k-p)(k-q)<0$

$\Rightarrow k \pm \sqrt{(k-p)(k-q)} = k \pm \sqrt{|k-p|\cdot|k-q|}\cdot i = z, \bar{z}$

one on upper half plane, one on lower half plane

distance^2 to the center $\frac{p+q}{2}$:

$|z - \frac{p+q}{2}|^2 = (k-\frac{p+q}{2})^2 + |(k-p)\cdot(k-q)|$

$= k^2 - (p+q)k + \frac{(p+q)^2}{4} - (k^2 - (p+q)k + pq)$

$= \frac{1}{4}\cdot(p^2 - 2pq + q^2) = \left|\frac{p-q}{2}\right|^2$

$\Rightarrow$ z is on the semicircle with center $\frac{p+q}{2}$, radius $\left|\frac{p-q}{2}\right|$

$\overset{f(z)=z}{\Longleftrightarrow}$ f has a fixed point z on the non-Euclidean line with ends p and q.