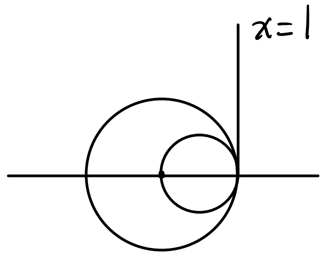
8.4.1 – 8.4.3



$x=1 \Leftrightarrow z+\bar{z}=2$

reflection in the unit circle

$$r\cdot e^{i\theta} \mapsto \frac{1}{r}e^{i\theta} = \frac{1}{re^{-i\theta}}$$

$$z \longmapsto \frac{1}{\bar{z}}$$

$$\left.\begin{array}{l}\frac{1}{\bar{z}}=w \Rightarrow z=\frac{1}{\bar{w}} \\ z+\bar{z}=2\end{array}\right\} \Rightarrow \frac{1}{\bar{w}}+\frac{1}{w}=2 \Leftrightarrow \boxed{w+\bar{w}=2w\bar{w}}$$

$$0=2w\cdot\bar{w}-w-\bar{w}=2\cdot\left(w\bar{w}-\tfrac{1}{2}w-\tfrac{1}{2}\bar{w}+\tfrac{1}{4}\right)-\tfrac{1}{2} \Leftrightarrow \underbrace{\left(w-\tfrac{1}{2}\right)\left(\bar{w}-\tfrac{1}{2}\right)}_{=\left|w-\frac{1}{2}\right|^2}=\frac{1}{4}$$

$$\Leftrightarrow \boxed{\left|w-\tfrac{1}{2}\right|=\tfrac{1}{2} \quad \text{center } \tfrac{1}{2}, \text{ radius}=\tfrac{1}{2}}$$

ends mapped to ends: $\boxed{\begin{array}{l} 1 \mapsto \frac{1}{\bar{1}}=1 \\ \infty \mapsto \frac{1}{\infty}=0 \end{array}}$

8.6.1 $\operatorname{dist}(y_1 i, y_2 i) = \left|\log \frac{y_2}{y_1}\right| = \left|\log y_2 - \log y_1\right|$, $y_1, y_2 > 0$.

Fix any $r>0$. Then for any $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$

$$\operatorname{dist}(r^k i, r^{k+1} i) = |\log r|, \quad \forall k \in \mathbb{Z}$$

8.6.2 Let $f(z) = \frac{az+b}{cz+d}$ with $ad-bc>0$

$$\begin{cases} f(0) = \frac{b}{d} = -1 \\ f(\infty) = \frac{a}{c} = 1 \end{cases} \Rightarrow f(z) = \frac{az+b}{az-b}$$

$ad-bc = a\cdot(-b) - b\cdot a = -2ab > 0$

choose $a=1$, $b=-1$:

$f(z) = \frac{z-1}{z+1}$ satisfy the condition.

8.6.3: $f(z) = \frac{z-1}{z+1}$ maps the y-axis to the unit semicircle.

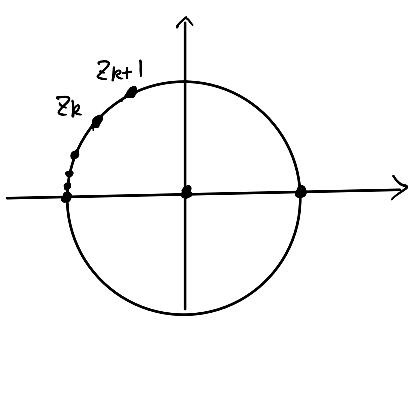
$$f(r^k i) = \frac{r^k i - 1}{r^k i + 1} = \frac{-(1-r^k i)^2}{(1+r^k i)(1-r^k i)} = \frac{-(1-r^{2k}-2r^k i)}{1+r^{2k}}$$

$$= \frac{(r^{2k}-1)+2r^k i}{1+r^{2k}} = z_k \in \text{unit semicircle.}$$

$$\operatorname{dist}(z_k, z_{k+1}) = \operatorname{dist}(r^k i, r^{k+1} i) = |\log r|$$

$$z_k = e^{i\theta_k} \Rightarrow \tan\theta_k = \frac{2r^k}{r^{2k}-1} \quad \left(\Rightarrow \tan\frac{\theta_k}{2} = -r^k \right.$$

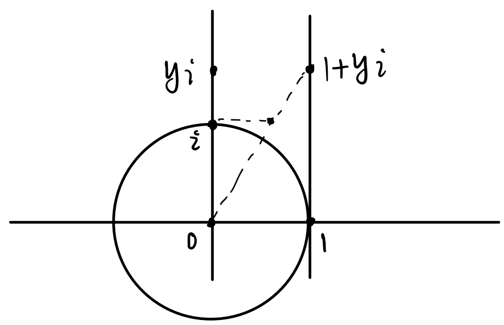
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$



8.6.4 $\quad \mathrm{dist}(yi,\ 1+yi) = \mathrm{dist}\left(\frac{yi}{y},\ \frac{1+yi}{y}\right) = \mathrm{dist}(i,\ y^{-1}+i)$

$\downarrow y \to +\infty$

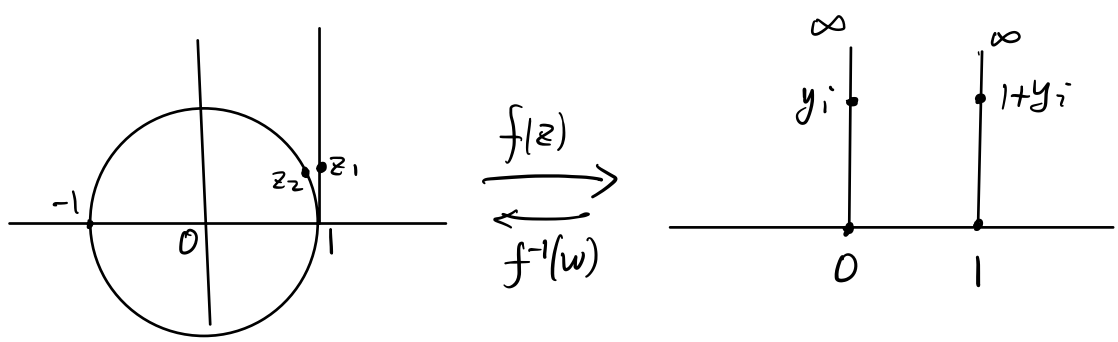
$\mathrm{dist}(i, i) = 0.$



8.6.5. $\quad f(z) = \frac{2}{1-z}$. Möbius transformation

$f(-1) = \frac{2}{2} = 1,\ f(1) = \infty$: unit circle $\longmapsto x = 1$.

$f(1) = \frac{2}{0} = \infty,\ f(\infty) = 0$: $x = 1 \longmapsto x = 0$.



$w = \frac{2}{1-z} \iff w - wz = 2 \iff z = \frac{w-2}{w} = 1 - \frac{2}{w} = f^{-1}(w)$

$z_1 = f^{-1}(yi) = \frac{yi-2}{yi} = 1 + \frac{2}{y}i, \quad f^{-1}(1+yi) = \frac{-1+yi}{1+yi} = \frac{y^2-1+2yi}{1+y^2} = z_2$

$\mathrm{dist}(z_1, z_2) = \mathrm{dist}(yi,\ 1+yi) \longrightarrow 0$ as $y \to \infty$ or equivalently

as $\frac{2}{y} \to 0$ and $\frac{2y}{1+y^2} \longrightarrow 0$