8.1.3 : $x^2+y^2=1$, $y>0$ to $(x-1)^2+y^2=4$

First rescale: $z \to 2\cdot z$: $x^2+y^2=4 \Leftrightarrow |\frac{z}{2}|^2=1$

Then translate: $z \mapsto z+1$: $(x-1)^2+y^2=4. \Leftrightarrow |z-1|^2=4$

8.1.4 : $x^2+y^2=1$, $y>0$ to any semicircle $(x-a)^2+y^2=r^2$

First rescale: $z \to r\cdot z$: $|\frac{z}{r}|^2=1 \Leftrightarrow |z|^2=r^2$

Then translate: $z \mapsto z+a$: $|z-a|^2=r^2 \Leftrightarrow (x-a)^2+y^2=r^2$.

$C_1=\{(x-a)^2+y^2=r^2\}_{y>0}$ to $\{(x-a')^2+y^2=r'^2\}_{y>0}=C_2$

$f_1: z \mapsto z-a$: $x^2+y^2=r^2$

$f_2: z \mapsto r^{-1}\cdot z$: $x^2+y^2=1$

$f_3: z \mapsto r'\cdot z$: $x^2+y^2=r'^2$

$f_4: z \mapsto z+a'$: $(x-a')^2+y^2=r'^2$

composition: $f_4 \circ f_3 \circ f_2 \circ f_1(z) = f_4 \circ f_3 \circ f_2(z-a) = f_4 \circ f_3(r^{-1}(z-a))$

$= f_4(r'\cdot r^{-1}(z-a)) = r'\cdot r^{-1}(z-a)+a'$.

transforms C_1 to C_2.

8.2.4 $\quad x \mapsto x+l \quad \rightsquigarrow \quad z \mapsto z+l$

8.2.6. $\quad x \mapsto kx \quad \rightsquigarrow \quad \begin{cases} z \mapsto kz & \text{when } k>0 \\ z \mapsto k\bar{z} & \text{when } k<0. \end{cases}$

$x \mapsto \frac{1}{x} \quad \rightsquigarrow \quad z \mapsto \frac{1}{\bar{z}}$

So: $\quad x \mapsto \frac{k}{x} \quad \rightsquigarrow \quad \begin{cases} z \mapsto \frac{k}{\bar{z}} & \text{when } k>0 \\ z \mapsto k \cdot \overline{\left(\frac{1}{\bar{z}}\right)} = \frac{k}{z} & \text{when } k<0. \end{cases}$

$$x \mapsto \frac{ax+b}{cx+d} = \frac{a}{c} + \frac{(ax+b)c - a\cdot(cx+d)}{c(cx+d)} = \frac{a}{c} - (ad-bc)\cdot\frac{1}{c^2\cdot(x+\frac{d}{c})}$$

$$f(z) = \frac{az+b}{cz+d} = \frac{a}{c} - (ad-bc)\cdot\frac{1}{c^2(z+\frac{d}{c})} \quad \text{extend} \quad x \mapsto \frac{ax+b}{cx+d} = f(x)$$

$$\| $$

$$\frac{a}{c} + \frac{-(ad-bc)}{c^2(z+\frac{d}{c})} \quad \text{if and only if} \quad ad-bc>0.$$

Similarly

$$f(\bar{z}) = \frac{a\bar{z}+b}{c\bar{z}+d} = \frac{a}{c} + \frac{-(ad-bc)}{c^2\overline{(z+\frac{d}{c})}} \quad \text{extend } f(x) \text{ if and only if}$$

$$-(ad-bc)>0 \quad \text{i.e.} \quad ad-bc<0.$$

8.2.5: $f_1(x) = \frac{a_1 x + b_1}{c_1 x + d_1}$, $f_2(x) = \frac{a_2 x + b_2}{c_2 x + d_2}$

$$f_3(x) = f_1 \circ f_2(x) = \frac{a_3 x + b_3}{c_3 x + d_3} \quad \text{where} \quad \underset{\substack{\shortparallel \\ A_3}}{\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}} = \underset{\substack{\shortparallel \\ A_1}}{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}} \underset{\substack{\shortparallel \\ A_2}}{\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}}.$$

$$\det(A_3) = \det(A_1)\det(A_2)$$

$\det(A_1) \cdot \det(A_2) > 0 \Rightarrow$

$$f_3(z) = \frac{a_3 z + b_3}{c_3 z + d_3} = \begin{cases} f_1(f_2(z)), & \text{if } \det A_1 > 0, \det A_2 > 0 \\ f_1(\overline{f_2(\bar{z})}) & \text{if } \det A_1 < 0, \det A_2 < 0. \end{cases}$$

$\det(A_1) \cdot \det(A_2) < 0 \Rightarrow$

$$f_3(\bar{z}) = \frac{a_3 \bar{z} + b_3}{c_3 \bar{z} + d_3} = \begin{cases} f_1(f_2(\bar{z})) & \text{if } \det A_1 > 0. \ \det A_2 < 0 \\ f_1(\overline{f_2(z)}) & \text{if } \det A_1 < 0, \det A_2 > 0. \end{cases}$$

$\Rightarrow$ The extension of $f_3(x)$ is always the composition of the extensions of $f_1(x)$ and $f_2(x)$.

8.3.1 $x \mapsto 2a - x$

8.3.2: $x \mapsto \frac{c^2}{x} = f(x)$ satisfies $f \circ f(x) = x$

and fixes only $\{-c, c\}$.

8.3.3: reflection in the point-pair $\{a, b\}$

Step 1: translate $\{a, b\}$ to have center at 0:

$$x \mapsto x - \frac{a+b}{2} = f_1(x)$$

$$\{a, b\} \mapsto \left\{\frac{a-b}{2}, -\frac{a-b}{2}\right\}$$

Step 2: reflect in the point pair $\left\{\frac{a-b}{2}, -\frac{a-b}{2}\right\}$

$$x \mapsto \frac{(a-b)^2}{4} \frac{1}{x}. = f_2(x)$$

Step 3: translate by $\frac{a+b}{2}$:

$$x \mapsto x + \frac{a+b}{2} = f_3(x)$$

Composition: $f(x) = f_3 \circ f_2 \circ f_1 (x)$

$$f_3(x) = f_3 \circ f_2 \circ f_1(x) = f_3 \circ f_2\left(x - \frac{a+b}{2}\right)$$

$$= f_3\left(\frac{(a-b)^2}{4} \cdot \frac{1}{x - \frac{a+b}{2}} \right) = \frac{(a-b)^2}{4} \cdot \frac{1}{x - \frac{a+b}{2}} + \frac{a+b}{2}$$

$$= \frac{(a-b)^2}{2 \cdot (2x - (a+b))} + \frac{a+b}{2} = \frac{(a-b)^2 + (a+b) \cdot (2x - (a+b))}{2\,(2x - (a+b))}$$

$$= \frac{2x \cdot (a+b) - 4ab}{2\,(2x - (a+b))} = \frac{x \cdot (a+b) - 2ab}{2x - (a+b)}$$

8.3.4. As $b \to \infty$,

$$\frac{x \cdot (a+b) - 2ab}{2x - (a+b)} = \frac{b \cdot (x - 2a) + x \cdot a}{-b + (2x - a)} \xrightarrow{b \to \infty} \frac{x - 2a}{-1} = 2a - x$$

reflection in $x = a$.

✓