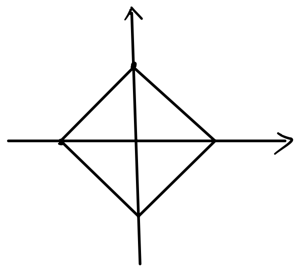
7.6.1



rotations: $\{1,\ i,\ -1,\ i\}$.

7.6.2.

$$\left\{ e^{i\cdot\frac{2\pi k}{n}},\ k=0,1,\cdots,n-1 \right\} = C_n \cong \mathbb{Z}/n\mathbb{Z}$$

$$\begin{gathered} e^{i\cdot\frac{2\pi k}{n}} \\ \parallel \\ \cos\frac{2\pi k}{n} + i\cdot\sin\frac{2\pi k}{n} \end{gathered}$$

7.7.1.

$$\left|\pm\tfrac{1}{2}\pm\tfrac{1}{2}i\pm\tfrac{1}{2}j\pm\tfrac{1}{2}k\right|^2 = \tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4} = 1 = 1^2.$$

7.7.2.

vertices of 4-dim cube.

$$\text{length side} = 1 = \left|\left(\tfrac{1}{2}+\tfrac{1}{2}i+\tfrac{1}{2}j+\tfrac{1}{2}k\right)-\left(-\tfrac{1}{2}+\tfrac{1}{2}i+\tfrac{1}{2}j+\tfrac{1}{2}k\right)\right|$$

7.7.3

$$\left|\left(\tfrac{1}{2}+\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}\right)-1\right| = \left|-\tfrac{1}{2}+\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}\right| = 1$$

$$= \left|\left(\tfrac{1}{2}+\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}\right)-i\right| = \left|\left(\tfrac{1}{2}+\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}\right)-j\right|$$

$$= \left|\left(\tfrac{1}{2}+\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}\right)-k\right|$$

$$\left|\left(\tfrac{1}{2}-\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}\right)-1\right| = \left|\left(\tfrac{1}{2}-\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}\right)-(-i)\right| = 1$$

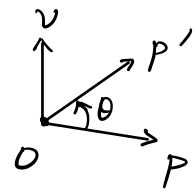
$$= \left|\left(\tfrac{1}{2}-\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}\right)-j\right|$$

$$= \left|\left(\tfrac{1}{2}-\tfrac{i}{2}+\tfrac{j}{2}+\tfrac{k}{2}\right)-k\right|$$

I. Use cross-product to find the axis:

$$v = \overrightarrow{OP} \times \overrightarrow{OP'} = \begin{vmatrix} i & j & k \\ a & b & c \\ a' & b' & c' \end{vmatrix}, \quad li + mj + nk = \frac{v}{|v|}.$$

Use cross-product to find the angle



$$\cos\theta = \frac{\overrightarrow{OP} \cdot \overrightarrow{OP'}}{|\overrightarrow{OP}| \cdot |\overrightarrow{OP'}|} = \overrightarrow{OP} \cdot \overrightarrow{OP'}, \quad 0 \le \theta \le \pi$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}, \quad \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}, \quad 0 \le \frac{\theta}{2} \le \frac{\pi}{2}.$$

(i) $$v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{vmatrix} = -\frac{2}{3} j + \frac{2}{3} k = \frac{2}{3}(-j+k)$$

axis: $$li + mj + nk = \frac{v}{|v|} = \frac{-j+k}{|-j+k|} = \frac{1}{\sqrt{2}}(-j+k).$$

$$\cos\theta = (1,0,0) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{3},$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\frac{1}{3}}{2}} = \sqrt{\frac{2}{3}}, \quad \sin\frac{\theta}{2} = \sqrt{\frac{1-\frac{1}{3}}{2}} = \sqrt{\frac{1}{3}}.$$

$$\Rightarrow q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} \cdot (li + mj + nk) = \frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}}(-j+k)$$
$$= \frac{1}{\sqrt{6}} \cdot (2 - j + k).$$

$-q = \frac{1}{\sqrt{6}}(-2 + j - k)$ also satisfies $f_{-q} = f_q$.

verify: $$q \cdot p \cdot q^{-1} = \frac{1}{\sqrt{6}}(2 - j + k) \cdot i \cdot \frac{1}{\sqrt{6}}(2 + j - k) = \frac{1}{6}(2i + k + j) \cdot (2 + j - k)$$
$$= \frac{1}{6}\left((-1+1) + i \cdot (4 - 1 - 1) + j(2+2) + k \cdot (2+2)\right) = \frac{1}{3} i + \frac{2}{3} j + \frac{2}{3} k.$$

(ii) $v = \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{vmatrix} = 0\cdot i + \left(-\frac{1}{3\sqrt{3}}\right)j + \frac{1}{3\sqrt{3}}k$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad \frac{1}{3\sqrt{3}}(-j+k)$

$\Rightarrow$ axis $\quad li + mj + nk = \frac{1}{\sqrt{2}}(-j+k)$

$\cos\theta = \left(\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{3\sqrt{3}}\cdot 5$

$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\frac{5}{3\sqrt{3}}}{2}} = \sqrt{\frac{3\sqrt{3}+5}{6\sqrt{3}}}$

$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{5}{3\sqrt{3}}}{2}} = \sqrt{\frac{3\sqrt{3}-5}{6\sqrt{3}}}$

$\Rightarrow q = \sqrt{\frac{3\sqrt{3}+5}{6\sqrt{3}}} + \sqrt{\frac{3\sqrt{3}-5}{6\sqrt{3}}}\cdot\frac{1}{\sqrt{2}}(-j+k)$

II. • Identity transformation: ± 1; $f_{\pm 1} = Id$

• Rotations around i-axis by angle $\frac{\pi}{2}$:

$$\cos\frac{\pi}{4} + \sin\frac{\pi}{4}(i) = \frac{1}{\sqrt{2}}(1+i), \quad -\frac{1}{\sqrt{2}}(1+i).$$

by angle π: $\cos\frac{\pi}{2} + \sin\frac{\pi}{2}(i) = i, \quad -i$

$\frac{3\pi}{2}$: $\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4} i = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i, \quad \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

Similarly. around j-axis: $\pm\frac{1}{\sqrt{2}}(1+j)$, $\pm j$, $\pm\frac{1}{\sqrt{2}}(1-j)$

k-axis: $\pm\frac{1}{\sqrt{2}}(1+k)$, $\pm k$, $\pm\frac{1}{\sqrt{2}}(1-k)$.

• Rotation around the diagonal axis $\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k$ by $\frac{2\pi}{3}$:

$$\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot\left(\left(\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k\right)\cdot\frac{2}{\sqrt{3}}\right) = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$$

$$\rightsquigarrow \quad \pm\left(\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k\right).$$

by $\frac{4\pi}{3}$: $\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}\left(\left(\frac{1}{2}i+\frac{1}{2}j+\frac{k}{2}\right)\frac{2}{\sqrt{3}}\right) = -\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$

$$\rightsquigarrow \quad \pm\left(-\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k\right) = \mp\left(\frac{1}{2}-\frac{1}{2}i-\frac{1}{2}j-\frac{1}{2}k\right)$$

Similarly, around the diagonal axis $\frac{1}{2}i-\frac{1}{2}j+\frac{1}{2}k$:

$$\pm\left(\frac{1}{2}+\frac{1}{2}i-\frac{1}{2}j+\frac{1}{2}k\right), \quad \pm\left(-\frac{1}{2}+\frac{1}{2}i-\frac{1}{2}j+\frac{1}{2}k\right).$$

around the diagonal axis $\frac{1}{2}i+\frac{1}{2}j-\frac{1}{2}k$:

$$\pm\left(\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j-\frac{1}{2}k\right), \quad \pm\left(-\frac{1}{2}+\frac{1}{2}i+\frac{1}{2}j-\frac{1}{2}k\right)$$

around the diagonal axis: $-\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k$:

$\pm\left(\frac{1}{2}-\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k\right)$, $\pm\left(-\frac{1}{2}-\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k\right)$.

- Rotation around the axis $\frac{1}{2}j+\frac{1}{2}k$ by angle π:

$$\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\cdot\left(\frac{1}{2}j+\frac{1}{2}k\right)\cdot\frac{2}{\sqrt{2}}=\frac{1}{\sqrt{2}}(j+k)\ ,\ -\frac{1}{\sqrt{2}}(j+k)$$

similarly. around the axis $\frac{1}{2}(j-k)$: $\frac{1}{\sqrt{2}}(j-k)$, $-\frac{1}{\sqrt{2}}(j-k)$

the axis $\frac{1}{2}i+\frac{1}{2}k$: $\pm\frac{1}{\sqrt{2}}(i+k)$

the axis $\frac{1}{2}i-\frac{1}{2}k$: $\pm\frac{1}{\sqrt{2}}(i-k)$

the axis $\frac{1}{2}i+\frac{1}{2}j$: $\pm\frac{1}{\sqrt{2}}(i+j)$.

the axis $\frac{1}{2}i-\frac{1}{2}j$: $\pm\frac{1}{\sqrt{2}}(i-j)$.

Altogether: ± 1, $\pm i$, $\pm j$, $\pm k$ — 8

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$\frac{1}{\sqrt{2}}(\pm 1\pm i)$, $\frac{1}{\sqrt{2}}(\pm 1\pm j)$, $\frac{1}{\sqrt{2}}(\pm 1\pm k)$ — $4\cdot 3=12$

$+$

$\pm\frac{1}{2}\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k$ — $2^4=16$

$+$

$\frac{1}{\sqrt{2}}(\pm j\pm k)$, $\frac{1}{\sqrt{2}}(\pm i\pm k)$, $\frac{1}{\sqrt{2}}(\pm i\pm j)$ — $4\cdot 3=12$

total 48