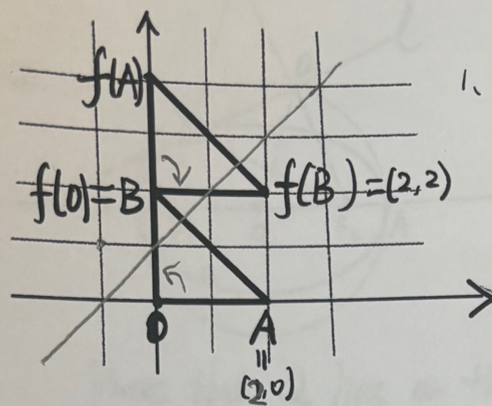

1. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry that maps ΔOAB to $\Delta f(O)f(A)f(B)$

(a) Explain why f must be a glide reflection.

(b) Find the axis and translation vector for the glide reflection f.

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(a) Isometries of $\mathbb{R}^2$ are translations, rotations and glide reflections. Only glide reflections reverse the orientation. Because f reverses the orientation, f must be a glide reflection.

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(b) The middle point of $\overline{O f(O)}$: $\frac{1}{2}((0,0)+(0,2)) = (0,1)$

middle point of $\overline{A f(A)}$: $\frac{1}{2}((2,0)+(0,4)) = (1,2)$ 5

middle point of $\overline{B f(B)}$: $\frac{1}{2}((0,2)+(2,2)) = (1,2)$.

The axis passes through these middle points. It is given by:

5

$$\ell: \quad (x(t), y(t)) = (0,1) + t\cdot(1,1) = (t, 1+t) \Leftrightarrow y = x+1.$$

translation vector: $v = \overrightarrow{r_\ell(O) f(O)} = (0,2) - (-1,1) = (1,1)$

$= \overrightarrow{r_\ell(A) f(A)} = (0,4) - (-1,3) = (1,1)$ 5

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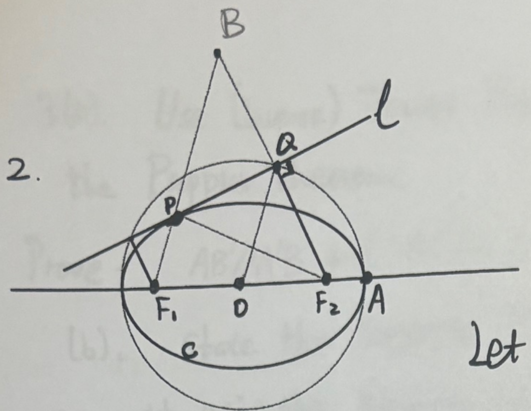
$= \overrightarrow{r_\ell(B) f(B)} = (2,2) - (1,1) = (1,1)$.

2. 

Let C be an ellipse with the Foci F_1 and F_2. Let ℓ be the tangent line to C at $P \in C$.

Let F_2Q be a line perpendicular to ℓ.

Prove that Q lies on the circle with radius equal to $|OA| = a$

(Assume $C = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$. so $|PF_1| + |PF_2| = 2a$)

In other words, prove $|OQ| = |OA|$.

<u>Proof</u>: Extend F_1P and F_2Q so that they intersect at B.

ℓ is tangent to C at $P \Rightarrow \angle BPQ = \angle F_2PQ$ 5

By assumption: $\angle PQB = \angle PQF_2 = \frac{\pi}{2}$.

$\Rightarrow \angle PBQ = \angle PF_2Q$.

ASA $\Rightarrow \triangle BPQ \cong \triangle F_2PQ \Rightarrow |F_2P| = |BP|, |F_2Q| = |BQ|$.

($|PQ| = |PQ|$) 5

5

$$\Rightarrow \frac{|F_2O|}{|F_2F_1|} = \frac{1}{2} = \frac{|F_2Q|}{|F_2B|} \xRightarrow[\text{Thales}]{\text{Inverse}} OQ \parallel F_1B \text{ and } \frac{|OQ|}{|F_1B|} = \frac{1}{2}$$ 5

$$\Rightarrow |OQ| = \frac{1}{2}|F_1B| = \frac{1}{2} \cdot (|F_1P| + |PB|) = \frac{1}{2}(|F_1P| + |PF_2|) = \frac{1}{2} \cdot 2a$$

$$= a = |OA|$$

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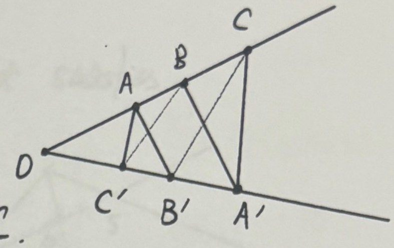
3(a). Use (Inverse) Thales Theorem to prove the Pappus Theorem:



Prove: $AB' // A'B$ and $AC' // A'C \Rightarrow BC' // B'C$.

(b). State the Projective Pappus Theorem and explain how to prove it using the projection and the above (original) Pappus Theorem.

3(a) $AB' // A'B \xRightarrow{\text{Thales}} \dfrac{|OA|}{|OB|} = \dfrac{|OB'|}{|OA'|}$ 5

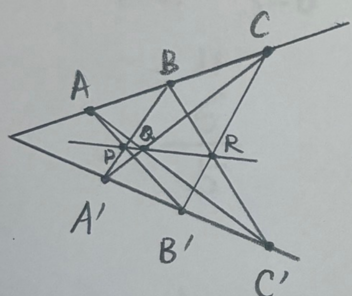
$AC' // A'C \Rightarrow \dfrac{|OA|}{|OC|} = \dfrac{|OC'|}{|OA'|}$

$\xRightarrow{\text{divide}} \dfrac{|OC|}{|OB|} = \dfrac{|OB'|}{|OC'|}$ 5

$\Downarrow$ Inverse Thales

$BC' // B'C$. 5

(b)



Projective Pappus: $AB' \cap A'B = P$, $AC' \cap A'C = Q$ 5

$BC' \cap B'C = R$. Then P, Q, R lie on the same line.

Proof: Use a projection $f: \mathbb{RP}^2 \to \mathbb{RP}^2$ to transform the line PQ to the horizon l_∞ (line at infinity). Then $f(P), f(Q) \in l_\infty$ 5

$f(P) = f(A)f(B') \cap f(A')f(B)$

$\Rightarrow f(A)f(B') // f(A')f(B)$

$f(A)f(C') // f(A')f(C)$

$\xRightarrow{\text{Pappus}} f(B)f(C') // f(B')f(C) \Rightarrow f(R) = f(B)f(C') \cap f(B')f(C) \in l_\infty$

$\Rightarrow R \in f^{-1}(l_\infty) = \text{line } PQ$.

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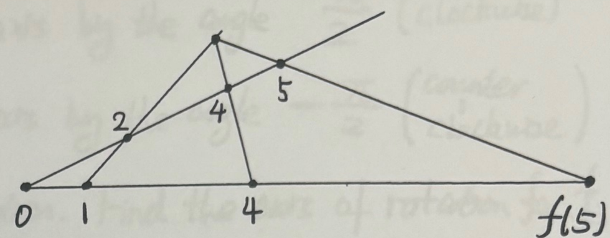
$\Rightarrow P, Q, R$ lie on the same line.

4. Let $f: \mathbb{RP}^1 \to \mathbb{RP}^1$ be a projection that satisfies

$f(0)=0,\ f(2)=1,\ f(4)=4.$

Find $f(5)$.

(Hint: projection preserves cross-ratios)



Set $f(5)=x$. Then:

$$[0,2,4,5] = [0,1,4,x] \quad 5$$

$$\begin{array}{ccc} \| & & \| \\ \dfrac{4-0}{4-2}\cdot\dfrac{5-2}{5-0} & & \dfrac{4-0}{4-1}\cdot\dfrac{x-1}{x-0} \\ \| & & \| \\ \dfrac{6}{5} & = & \dfrac{4(x-1)}{3x} \end{array} \quad 5 \quad 5$$

$$\Rightarrow 18x = 20x-20 \Leftrightarrow 2x = 20 \Rightarrow x = 10 = f(5).$$

5 5

5. Let f_1, f_2 be two rotations of S^2:

f_1 : rotation around the i-axis by the angle $\frac{\pi}{2}$ (clockwise)

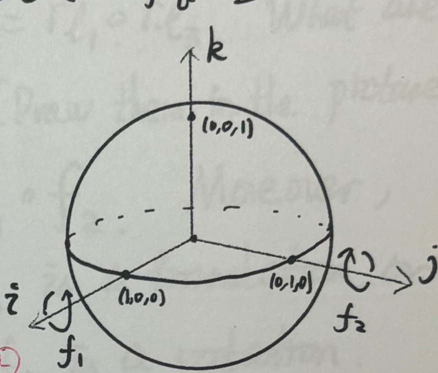
f_2 : rotation around the j-axis by the angle $-\frac{\pi}{2}$ (counter clockwise)

Set $f_3 = f_1 \circ f_2$ which is also a rotation. Find the axis of rotation for f_3.

(Hint: • rotation around $li+mj+nk$ by angle $\theta \leftrightarrow f_q : S^2 \to S^2$ with

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(li+mj+nk)$$

• $f_{q_1} \circ f_{q_2} = f_{q_1 \cdot q_2}$)



$$q_1 = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} i = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \quad 5$$

$$q_2 = \cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) j = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} j \quad 5$$

$$q_1 \cdot q_2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) \cdot \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} j\right) = \frac{1}{2} + \frac{1}{2} i - \frac{1}{2} j - \frac{1}{2} k \quad 5$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{3}} (i - j - k). \quad 5$$

$\Rightarrow$ axis : $\frac{1}{\sqrt{3}}(i-j-k)$. 5

(angle θ : $\cos\frac{\theta}{2} = \frac{1}{2}$, $\sin\frac{\theta}{2} = \frac{\sqrt{3}}{2} \Rightarrow \frac{\theta}{2} = \frac{\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$)

6. Let f_1, f_2 be two rotations of $\mathbb{R}^2$.

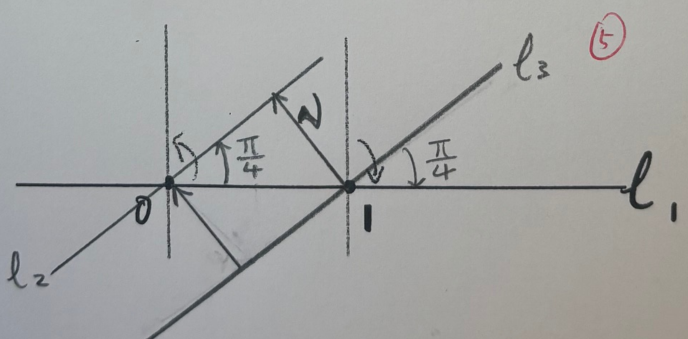
f_1 is the rotation around $(0,0)$ by angle $\frac{\pi}{2}$

f_2 is the rotation around $(1,0)$ by angle $-\frac{\pi}{2}$.

Set $l_1 = x$-axis, r_{l_1} = the reflection in l_1.

(a) If we write $f_1 = r_{l_2} \circ r_{l_1}$ and $f_2 = r_{l_1} \circ r_{l_3}$. What are the lines (Mirrors) l_2 and l_3? (Draw them in the picture)

(b) Classify the composition $f_3 = f_1 \circ f_2$. Moreover, Find the translation vector if f_3 is a translation, or find its center and angle if f_3 is a rotation.

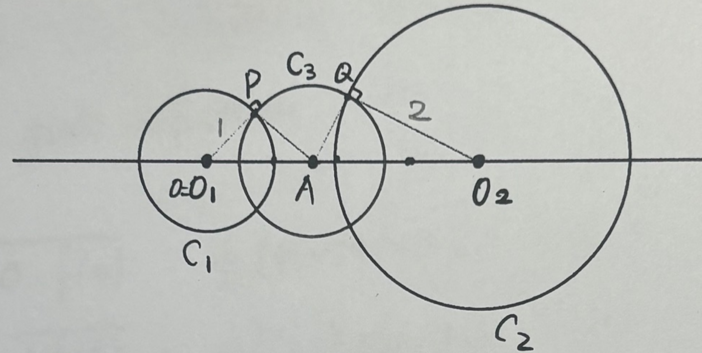


$f_3 = f_1 \circ f_2 = r_{l_2} \circ r_{l_1} \circ r_{l_1} \circ r_{l_3} = r_{l_2} \circ Id \circ r_{l_3} = r_{l_2} \circ r_{l_3}$ 5

$l_2 // l_3 \Rightarrow f_3$ is a translation 5

Translation vector $= 2 \cdot v = 2 \cdot (-\frac{1}{2}, +\frac{1}{2}) = (-1, 1)$. 5

8. Let $C_1 = \{|z|=1\}$, $C_2 = \{|z-4|=2\}$ be two non-Euclidean lines. Let r_{C_1}, r_{C_2} be reflections across the C_1, C_2 respectively. Let $f = r_{C_2} \circ r_{C_1}$ be the composition that is a non-Euclidean translation.



Find the "axis" of translation for f, which is the semi-circle C_3 centered at A on x-axis and orthogonal to both C_1 and C_2.

(Calculate the center A and the radius $r = |AP| = |AQ|$)

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Let $A = (x_0, 0)$. Then $|AO_1|^2 - |O_1P|^2 = |oP|^2 = |oQ|^2 = |AO_2|^2 - |O_2Q|^2$

$\Rightarrow x_0^2 - 1 = (4-x_0)^2 - 4 = 16 - 8x_0 + x_0^2 - 4 = x_0^2 - 8x_0 + 12$

5

$\Leftrightarrow 8x_0 = 13 \Rightarrow x_0 = \frac{13}{8}$. 5 5

$|oP|^2 = |AO_1|^2 - |O_1P|^2 = \left(\frac{13}{8}\right)^2 - 1^2 = \frac{169}{64} - 1 = \frac{105}{64}$ 5

$\left(= |AO_2|^2 - |O_2Q|^2 = \left(4 - \frac{13}{8}\right)^2 - 4 = \frac{(19)^2}{64} - 4 = \frac{361-256}{64} = \frac{105}{64} \checkmark\right)$

Radius $= |oP| = \frac{\sqrt{105}}{8}$