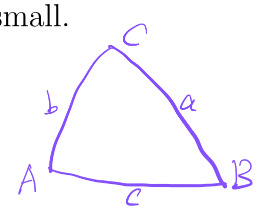
!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(20pts) (i) State both the law of cosine for sides and the law of cosine for angles for spherical triangles.
(ii) Explain how the first one is related to the law of cosine on the Euclidean plane when the sides are small.



(i) Law of cosine for sides: $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cos \angle C$ 5

for angles: $\cos \angle C = -\cos A \cdot \cos B + \sin A \cdot \sin B \cdot \cos c$. 5

(ii) When the sides a, b, c are small, use $\cos x = 1 - \frac{x^2}{2} + o(|x|^2)$ to get

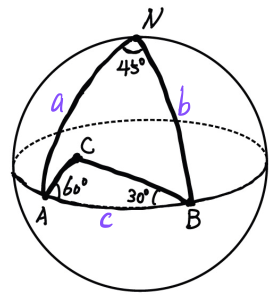
and $\sin x = x + o(|x|)$ 5

$$1 - \frac{c^2}{2} = \left(1 - \frac{a^2}{2}\right) \cdot \left(1 - \frac{b^2}{2}\right) + a \cdot b \cdot \cos \angle C$$ 5

$$= 1 - \frac{a^2 + b^2}{2} + o(|a|^2 + |b|^2) + ab \cdot \cos \angle C$$

$\Rightarrow \quad c^2 = a^2 + b^2 - 2ab \cdot \cos \angle C$ law of cosine on Euclidean plane

2(20pts) In the following picture on a sphere, N is the north pole and A, B are on the equator. First **calculate** $\cos(C)$. Is the angle C bigger than or smaller than $90°$? Explain your reason.



Use the law of cosine for sides to get

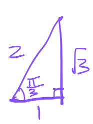
$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos \angle ANB$$ 9

$$= \cos\frac{\pi}{2} \cdot \cos\frac{\pi}{2} + \sin\frac{\pi}{2} \sin\frac{\pi}{2} \cos\frac{\pi}{4} = 0 \cdot 0 + 1 \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

Use the law of cosine for angles to get:

$$\cos\angle C = -\cos\angle CAB \cdot \cos\angle CBA + \sin\angle CAB \cdot \sin\angle CBA \cdot \cos c$$ 9

$$= -\cos\frac{\pi}{3} \cdot \cos\frac{\pi}{6} + \sin\frac{\pi}{3} \cdot \sin\frac{\pi}{6} \cdot \frac{\sqrt{2}}{2}$$



$$= -\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{3}}{4}\left(-1 + \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{6}}{8} - \frac{\sqrt{3}}{4} < 0$$

$$\Rightarrow \angle C > \frac{\pi}{2} = 90^0$$ 2

Or: $\angle CAB + \angle CBA + \angle C - \pi = |\triangle ABC| > 0$

$$\begin{array}{c} \angle CAB + \angle CBA + \angle C - \pi \\ \| \\ \frac{\pi}{3} + \frac{\pi}{6} + \angle C - \pi \\ \| \\ \angle C - \frac{\pi}{2} \end{array}$$

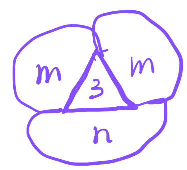
$$\Rightarrow \angle C > \frac{\pi}{2}$$

3(20 pts) For a semiregular tiling on the Euclidean plane, there are three tiles at a vertex. One of the tile is an equilateral triangle. What regular polygons are the other two tiles? Explain your reason.

Assume they are regular ℓ-gon, m-gon and n-gon. Then:

$$\frac{\ell-2}{\ell}\cdot\pi + \frac{m-2}{m}\cdot\pi + \frac{n-2}{n}\pi = 2\pi$$ 5

$$\Rightarrow \frac{2}{\ell} + \frac{2}{m} + \frac{2}{n} = 1.$$ 5

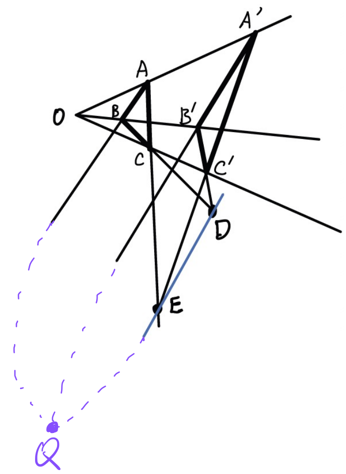


If ℓ is odd, then $m=n$. 5

$\ell = 3 \Rightarrow \frac{2}{3} + \frac{2}{m} + \frac{2}{m} = 1 \Rightarrow \frac{4}{m} = \frac{1}{3} \Rightarrow m = n = 12.$ 5

So the other two tiles are regular 12-gons = dodecagons.

4(20pts) In the following picture, assume that AB is parallel to $A'B'$. By using Desargues' theorem, what can you say about the line connecting D and E? Explain your reason.



$\triangle ABC$ and $\triangle A'B'C'$ are in perspective

By Desargues' Thm, D, E and $AB \cap A'B'$ lie on the same line. 5

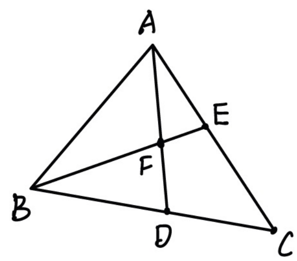
$AB \parallel A'B' \Rightarrow$ the point $\overset{Q}{\overset{\parallel}{AB \cap A'B'}}$ lies on the line at infinity 5

and represents the direction of the lines AB and $A'B'$

$\Rightarrow$ Q lies on the line DE which means Q represents the direction of DE 5

$\Rightarrow$ $DE \parallel AB \parallel A'B'$. 5

5(20pts) Draw the dual picture to the following picture on a projective plane.



Dual picture :