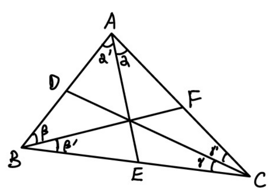
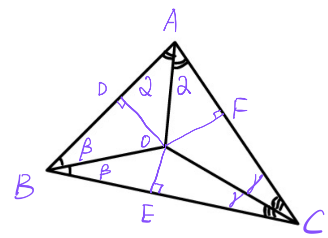
!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(20pts) **(i)** State Ceva's theorem in its usual sides form and trigonometric form.
(ii) Prove that the bisection lines of angles intersect at a common point.





(i) Ceva's Theorem: AE, BF, CD are concurrent i.e. intersect at a common point.

if and only if (iff) Sides form: $\frac{|AD|}{|DB|}\cdot\frac{|BE|}{|EC|}\cdot\frac{|CF|}{|FA|}=1$. 5

iff trigonometric form: $\frac{\sin\alpha}{\sin\alpha'}\cdot\frac{\sin\beta}{\sin\beta'}\cdot\frac{\sin\gamma}{\sin\gamma'}=1$. 5

(ii) Because $\frac{\sin\alpha}{\sin\alpha}\cdot\frac{\sin\beta}{\sin\beta}\cdot\frac{\sin\gamma}{\sin\gamma}=1$, by Ceva's Thm in its trigonometric form, the 3 bisection lines of angles intersect at a common point. 10

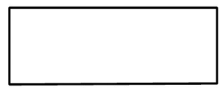
OR: Let AO, BO be bisection lines of $\angle A$ and $\angle B$ respectively.

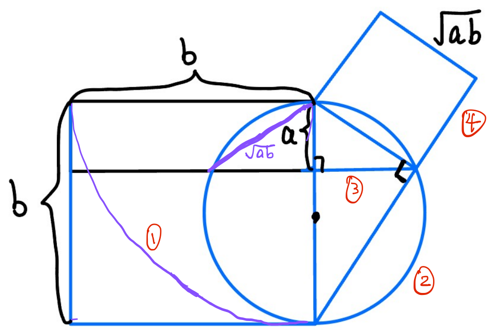
Assume they intersect at O.

By ASA, $\triangle ADO \cong AFO \Rightarrow |OD|=|OF|$
$\triangle BDO \cong \triangle BEO \Rightarrow |OD|=|OE|$
$\Big\} \Rightarrow |OE|=|OF| \xrightarrow[\text{Thm}]{\text{Pythagorean}} |CE|=|CF|$

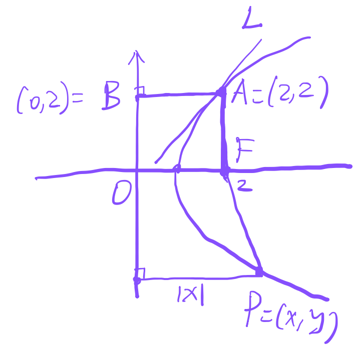
$\xrightarrow{SSS} \triangle OEC \cong \triangle OFC \Rightarrow \angle OCE = \angle OCF \Leftrightarrow$ OC is the bisection line of $\angle C$

2(20pts) Explain how to use compass/straightedge to construct a square that has the same area as the area of a given rectangle.





3(20 pts) Assume C is a parabola with its focus at $(2,0)$ and directrix $\ell : \{x = 0\}$. Find the equation for the parabola C. Then find the equation for its tangent line at the point $(2,2)$.



$P \in C \iff |PF| = \text{distance from } P \text{ to } \ell$

$\quad \| $

(x,y)

$$\iff \sqrt{(x-2)^2 + y^2} = |x| \qquad 5$$

$$\iff (x-2)^2 + y^2 = x^2$$

$$\| $$

$$x^2 - 4x + 4 + y^2$$

$$\iff y^2 = 4x - 4. \quad \text{equation for the parabola } C. \qquad 5$$

The tangent line L at $(2,2)$ is the bisection line of the angle $\angle BAF = 90^0$. 5

So its slope is given by $\tan 45^\circ = 1$. The equation is for L is:

$$y - 2 = 1 \cdot (x-2) \iff y = x. \qquad 5$$

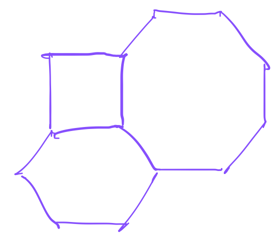
One can also use calculus: $y^2 = 4x - 4 \Rightarrow 2y \cdot \frac{dy}{dx} = 4$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{y} \overset{(x,y)=(2,2)}{=\!=\!=} \frac{2}{2} = 1 \Rightarrow \text{The tangent line is given by}$$

$$y - 2 = 1 \cdot (x-2) \iff y = x.$$

4(20pts) The truncated cuboctahedron has the vertex configuration (4, 6, 8). Find the number of squares, hexagons and octagons on its whole face.

$F_n =$ number of n-gons. $n = 4, 6, 8$.



total number of vertices: $V = 4F_4 = 6F_6 = 8F_8 \Rightarrow \begin{cases} F_6 = \frac{2}{3}F_4 \\ F_8 = \frac{1}{2}F_4 \end{cases}$ 5

edges: $E = \frac{4F_4 + 6F_6 + 8F_8}{2} = 2F_4 + 3F_6 + 4F_8$ 3

faces: $F = F_4 + F_6 + F_8$ 2

$\Rightarrow E = 2F_4 + 2F_4 + 2F_4 = 6F_4$

$F = F_4 + \frac{2}{3}F_4 + \frac{1}{2}F_4 = \frac{13}{6}F_4$

5

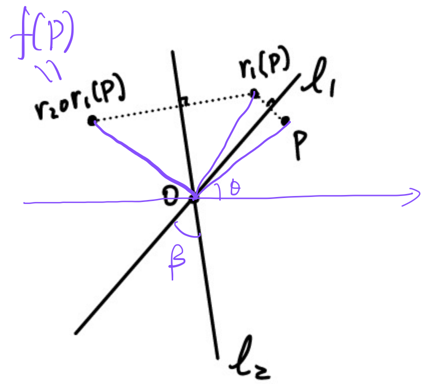
Euler characteristic $2 = \chi(S^2) = V - E + F = 4F_4 - 6F_4 + \frac{13}{6}F_4 = \frac{1}{6}F_4$

$\Rightarrow F_4 = 12$ 12 squares

$\Rightarrow F_6 = \frac{2}{3} \cdot 12 = 8$ 8 hexagons 5

$F_8 = \frac{1}{2}F_4 = 6$ 6 octagons

5(20pts) Assume that ℓ_1, ℓ_2 are two lines on the plane intersecting at O. Denote by r_1 and r_2 the reflections with respect to ℓ_1 and ℓ_2 respectively. Classify the isometry type of the composition $f = r_2 \circ r_1$. In other words, determine whether f is a translation, rotation, reflection or glide reflection and explain the reason.



f has a unique fixed point $O \Rightarrow f$ is a rotation.

The angle of rotation $\angle POf(P) = 2\beta$.

$$\theta \mapsto 2\alpha_1 - \theta \mapsto 2\alpha_2 - (2\alpha_1 - \theta) = 2(\alpha_2 - \alpha_1) + \theta = 2\beta + \theta.$$