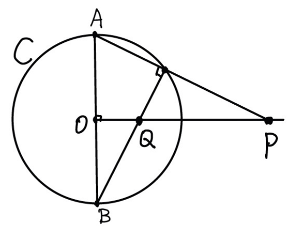
!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(10pts) (i) Explain what is an inversion of a point P with respect to a circle C.
(ii) In the following picture, O is the center of the circle C and $AB \perp OP$. Prove that Q is the inversion of P with respect to the circle C.



(i) $|OP|\cdot|OQ| = R^2$. $\quad \vec{OQ} = \lambda\cdot\vec{OP} \quad \lambda>0$

R= radius of C $\quad = \frac{R^2}{|OP|}\cdot\frac{\vec{OP}}{|OP|} = \frac{R^2}{|\vec{OP}|^2}\vec{OP}$.

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(ii) $\angle OBQ + \angle OAP = 90^\circ \Rightarrow \angle OBQ = \angle OBA$.

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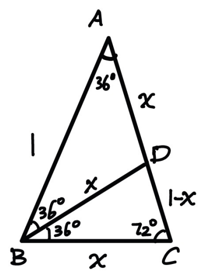
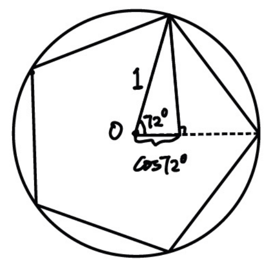
$\Rightarrow \triangle OBQ \sim \triangle OPA$

$\Rightarrow \frac{|OQ|}{|OA|} = \frac{|OB|}{|OP|} \Rightarrow |OQ|\cdot|OP| = |OA|\cdot|OB| = R^2$

2(10pts) (i) Calculate $\cos(72°)$ (Hint: use similar triangles $\Delta ABC \sim \Delta BCD$).
(ii) Explain why the regular pentagon is constructible by compass/straightedge.





(i) $\Delta ABC \sim \Delta BCD \Rightarrow \dfrac{|AB|}{|BC|} = \dfrac{|BC|}{|CD|} \Leftrightarrow \dfrac{1}{x} = \dfrac{x}{1-x}$

$\Leftrightarrow x^2 = 1-x \Leftrightarrow x^2+x-1=0 \Rightarrow x = \dfrac{-1 \pm \sqrt{1+4}}{2}$

$\Rightarrow x = \dfrac{-1+\sqrt{5}}{2} > 0 \Rightarrow \cos 72° = \dfrac{x}{2} = \dfrac{\sqrt{5}-1}{4}$. 5

(ii) regular n-gon is constructible $\Leftrightarrow \cos\frac{2\pi}{n}$ is constructible. 3

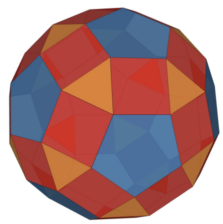
$\cos 72° = \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ is constructible because

a, b constructible ($b \neq 0$) $\Longrightarrow a \cdot b,\ \frac{a}{b},\ na,\ \sqrt{a}$ are constructible 2

n is any integer

So the pentagon = regular 5-gon is constructible (by compass/straigtedge)

3(10pts) The semiregular polyhedra rhombicosidodecahedron shown in the following has the vertex configuration $(3, 4, 5, 4)$. Calculate the number of triangles, squares and pentagons. Write the detailed calculation.



F_n = number of regular n-gons

$\Rightarrow$ number of vertices $V = 3 \cdot F_3 = \frac{4F_4}{2} = 5F_5$ 2

number of edges $E = (3F_3 + 4F_4 + 5F_5)/2$ 2

number of faces $F = F_3 + F_4 + F_5$ 2

$\Rightarrow F_4 = \frac{3}{2}F_3, \quad F_5 = \frac{3}{5}F_3$

$E = (3F_3 + 6F_3 + 3F_3)/2 = 6F_3$

$F = F_3 + \frac{3}{2}F_3 + \frac{3}{5}F_3 = \frac{10+15+6}{10}F_3 = \frac{31}{10}F_3$

Euler characteristic $= 2 = V - E + F = 3F_3 - 6F_3 + \frac{31}{10}F_3 = \frac{1}{10}F_3$ 2

$\Rightarrow F_3 = 20, \quad F_4 = \frac{3}{2} \cdot 20 = 30, \quad F_5 = \frac{3}{5} \cdot 20 = 12.$

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So 20 equilateral triangles. 30 squares and 12 pentagons.

4(10pts) A semiregular tiling of the Euclidean plane has 5 tiles at each vertex. Prove that there are at least 3 triangles. What are possible vertex configurations when there are exactly three triangles?

Proof: Suppose there are at most 2 triangles. Then the total angle at a vertex is at least

$$2\cdot 60+3\cdot 90=120+270=390>360^\circ$$

which is impossible.

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If there are 3 triangles, then the remaining angles at a vertex is $360-3\cdot 60=360-180=180^\circ$ which must be from 2 squares. So the vertex configuration is $(4,4,3,3,3)$ or $(4,3,4,3,3)$ or $(4,3,3,4,3)$ or $(4,3,3,3,4)$

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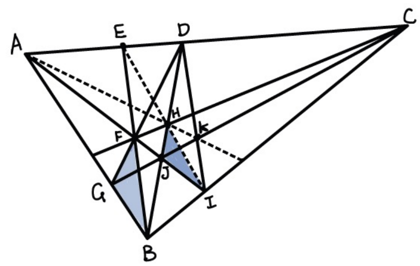
So there are two possibilities: $(4,4,3,3,3)=(3,3,3,4,4)$ 2

or $(4,3,4,3,3)=(3,3,4,3,4)$. 2

5(10pts) (i) In the following picture, explain how to use Desargues' Theorem for the two triangles ΔFGB and ΔHJI to conclude that the points B, F and E are on the same line.

(ii) Which two triangles can be used to apply Desargues' Theorem in order to prove that the three points A, H and K are on the same line.



(i) ΔFGB and ΔHJI are in perspective from C.

$\Rightarrow GB \cap JI = A$, $FG \cap HJ = D$ and $BF \cap IH$ are on the same line

$\Rightarrow BF \cap IH$ are on the line $AD \Leftrightarrow BF \cap AD = E$ and H, I are on the same line

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(ii) ΔFJB and ΔHKI are in perspective from C

$\Rightarrow BF \cap IH = E$, $BJ \cap IH = D$ and $JF \cap KH$ are on the same line

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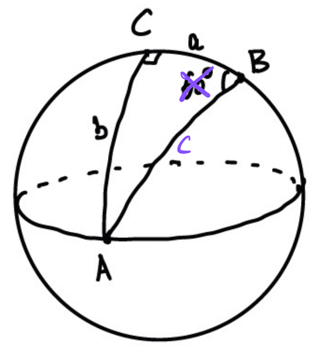
$\Rightarrow JF \cap DE = A$ and H, K are on the same line.

6(18pts) Assume $\widehat{ABC}$ is a right spherical triangle with right angle at C on a sphere with radius 1. Assume that C is the north pole and A is on the equator. Assume that the angle $\angle B$ is equal to $60^\circ = \frac{\pi}{3}$. Use spherical trigonometry to calculate the lengths $|\widehat{BC}|$, $|\widehat{AB}|$ and $\cos\angle A$. (Note: $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$)

This problem is defective.



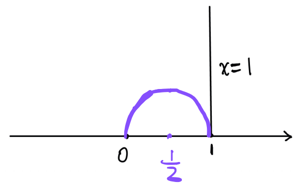
$$\sin B = \frac{\sin b}{\sin C}, \quad \cos B = \frac{\sin a}{\sin C}\cdot\cos b \Rightarrow \tan B = \frac{\sin b}{\sin a\cdot\cos b} = \frac{\tan b}{\sin a}$$

$$\Leftrightarrow \quad \sin a = \frac{\tan b}{\tan B} \quad \Leftrightarrow \quad \tan b = (\tan B)\cdot\sin a \Rightarrow |\tan b| \leq |\tan B|.$$

$$b = \frac{\pi}{2} \Rightarrow \tan b = \infty \Rightarrow \tan B = \infty \Leftrightarrow B = \frac{\pi}{2} \Leftarrow A \text{ is a pole w.r.t. } \widehat{BC}$$

7(16pts) Let T_γ be the isometry associated to the 2×2 matrix $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$.

(i) Is this T_γ a rotation or a translation?
(ii) What is the image of the geodesic $\{x = \mathrm{Re}(z) = 1\}$?



(i) $T_\gamma(z) = \dfrac{0\cdot z+1}{-1\cdot z+2} = \dfrac{1}{-z+2}$ 4

Find fixed points of T_γ: $\dfrac{1}{-z+2} = z \iff 1 = -z^2+2z$ 2

$\iff 0 = z^2 - 2z + 1 = (z-1)^2 \iff z = 1$.

$\Rightarrow$ there is only one fixed point $z=1$ on the $\mathbb{RP}^1 = \mathbb{R}\cup\{\infty\}$ 2

$\Rightarrow$ T_γ is a parabolic translation.

(ii) T_γ maps geodesics to geodesics. 2

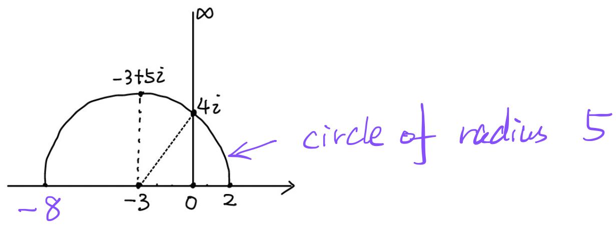
$T_\gamma(1) = \dfrac{1}{-1+2} = 1$, $T_\gamma(\infty) = \dfrac{1}{-\infty+2} = 0$. 2

$\Rightarrow$ T_γ maps $\{x=1\}$ to the geodesic with ends at 1 and 0.

which is the upper half circle that connects 1 and 0: 4

$$\left\{ \left|z-\tfrac{1}{2}\right| = \tfrac{1}{2},\ \mathrm{Im}\, z > 0 \right\}.$$

8(16pts) (i) Find an isometry T_γ of $\mathbb{H}$ that fixes $4i$ and sends ∞ to 2.
(ii) Calculate the hyperbolic distance from $P = -3+5i$ to $Q = 4i$.



(i) T_γ maps $4i \longmapsto 4i$

$\infty \longmapsto 2$

$0 \longmapsto -8$

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Verify: $\frac{2\cdot 4i-16}{4i+2} = \frac{8i-16}{4i+2} = 4i$

$\infty \longmapsto 2$

$0 \longmapsto -\frac{16}{2} = -8$

$$\Rightarrow (z,4i;\infty,0) = (w,4i;2,-8)$$

4

$$(z,4i;\infty,0) = \frac{z-\infty}{z-0} \Big/ \frac{4i-\infty}{4i-0} = \frac{z-\infty}{4i-\infty}\cdot\frac{4i}{z} = \frac{4i}{z}$$

$$(w,4i;2,-8) = \frac{w-2}{w+8} \Big/ \frac{4i-2}{4i+8} = \frac{2(w-2)}{i(w+8)}, \qquad \frac{4i-2}{4i+8} = \frac{i}{2}$$

$$\frac{4i}{z} = \frac{2(w-2)}{i(w+8)}$$

$$w(z) = \frac{2z-16}{z+2}$$

$\Updownarrow$

$$\Leftrightarrow -\cancel{4}_{2}(w+8) = \cancel{2}z(w-2) \Leftrightarrow -2w-16 = zw-2z \Leftrightarrow (z+2)w = 2z-16$$

4

$$\Rightarrow T_\gamma(z) = \frac{2z-16}{z+2}, \qquad \gamma = \begin{pmatrix} 2 & -16 \\ 1 & 2 \end{pmatrix}$$

(2) $$d(P,Q) = \left|\ln(\underset{-3+5i}{P},\underset{4i}{Q};2,-8)\right| = \ln\left|\frac{2(-3+5i-2)}{i(-3+5i+8)}\right| = \ln\left|\frac{2(-5+5i)}{(5+5i)}\right| = \ln 2.$$

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