Cross Ratio: $z_1, z_2, z_3, z_4 \in \mathbb{C}\cup\{\infty\}$

$$(z_1,z_2;z_3,z_4) = \frac{z_1-z_3}{z_1-z_4} \Big/ \frac{z_2-z_3}{z_2-z_4} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$$

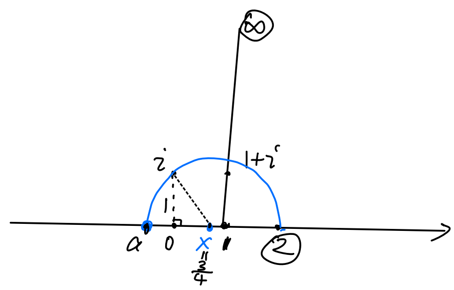
$$(z_1,z_2;\infty,z_4) = \frac{(z_1-\infty)(z_2-z_4)}{(z_1-z_4)(z_2-\infty)} = \underbrace{\frac{-\infty}{-\infty}}_{=1}\cdot\frac{z_2-z_4}{z_1-z_4} = \frac{z_2-z_4}{z_1-z_4}$$

$$(z_1,\infty;z_3,z_4) = \frac{z_1-z_3}{z_1-z_4} \Big/ \underbrace{\frac{\infty-z_3}{\infty-z_4}}_{=\frac{\infty}{\infty}=1} = \frac{z_1-z_3}{z_1-z_4}$$

$$T_\gamma(z) = \frac{az+b}{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$(z_1,z_2;z_3,z_4) = (T_\gamma(z_1), T_\gamma(z_2); T_\gamma(z_3), T_\gamma(z_4))$$

Ex: T_γ: $i+1 \to i$, $\infty \to 2$, $1 \mapsto -\frac{1}{2}$



$$(z, i+1; \infty, 1) = (\overset{w}{T_\gamma(z)}, i; 2, -\tfrac{1}{2})$$

$$\frac{z-\infty}{z-1}\Big/\frac{i+1-\infty}{i+1-1} = \frac{z-\infty}{-\infty}\cdot\frac{i}{z-1} = \frac{i}{z-1}$$

$$\frac{w-2}{w+\frac12}\Big/\frac{i-2}{i+\frac12} = \frac{w-2}{w+\frac12}\cdot\frac{i+\frac12}{i-2}, \qquad \frac{i+\frac12}{i-2} = \frac{2i+1}{2(i-2)} = -i$$

$$= -\frac{(w-2)\cdot i}{2w+1}$$

$|x-2| = \sqrt{x^2+1^2}$

$x^2-4x+4 = x^2+1 \Rightarrow 4x=3 \Rightarrow x=\frac{3}{4}$

$2\cdot\frac{3}{4} = a+2 \Rightarrow a = \frac{3}{2}-2 = -\frac{1}{2}$

$$\frac{2i+1}{i-2} = \frac{1+2i}{-2+i} = \frac{(1+2i)(-2-i)}{(-2+i)(-2-i)}$$

$$-i = \frac{-5i}{5} = \frac{(-2+2)+(-i-4i)}{5}$$

$$\frac{\cancel{\gamma}}{z-1} = -\frac{(w-2)\cancel{\gamma}}{2w+1} \Rightarrow -2w-1 = (z-1)(w-2) = zw - 2z - w + 2$$

$$\Rightarrow \underbrace{-1+2z-2}_{=\,2z-3} = \underline{z\cdot w - w + 2w} = zw + w = (z+1)w$$

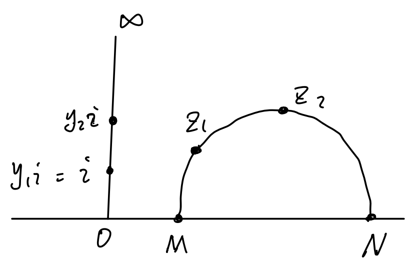
$$\Rightarrow w = \frac{2z-3}{z+1} = T_\gamma(z).$$

$$\gamma = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$$

$$T_\gamma(\infty) = \frac{2\infty - 3}{\infty + 1} = 2.$$

$$T_\gamma(1) = \frac{2\cdot 1 - 3}{1+1} = -\frac{1}{2}$$

$$T_\gamma(i+1) = \frac{2\cdot(i+1)-3}{i+1+1} = \frac{-1+2i}{2+i} = i \quad \checkmark$$



$z_1 \mapsto i$
$M \mapsto 0$
$N \mapsto \infty$

$$d(y_1 i, y_2 i) = \left|\ln \frac{y_2}{y_1}\right|$$

$$(z, z_1; M, N) = (T_\gamma(z), \underbrace{i}_{=\,T_\gamma(z_1)}; 0, \infty)$$

$$(z_2, z_1; M, N) = (T_\gamma(z_2), T_\gamma(z_1); 0, \infty).$$

$$= (y_2 i, i; 0, \infty)$$

$$= \frac{y_2 i - 0}{y_2 i - \infty} \Big/ \frac{i-0}{i-\infty} = \frac{y_2 i}{i} \cdot \frac{i-\infty}{y_2 i - \infty} = y_2$$

$$\frac{-\infty}{-\infty} = 1$$

$$\big|\ln(z_2, z_1; M, N)\big| = |\ln y_2| \Longleftarrow$$

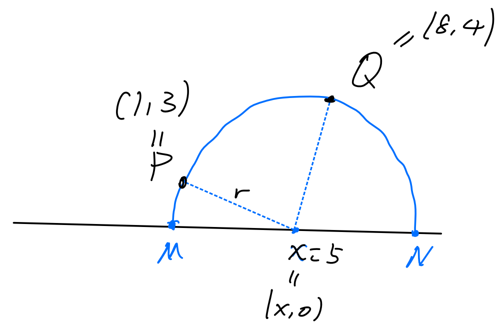
$$= |d(T_\gamma(z_1), T_\gamma(z_2))|$$

$$= d(z_1, z_2).$$

$$\Rightarrow \boxed{d(z_1, z_2) = \big|\ln(z_2, z_1; M, N)\big| = \big|\ln(z_1, z_2; M, N)\big|}$$

$$\left(|\ln x^{-1}| = |-\ln x| = |\ln x|\right)$$

Ex: $P = 1+3i$, $Q = 8+4i$



$$(x-1)^2+3^2 = (x-8)^2+4^2$$

$$\underbrace{}_{\cancel{x^2}-2x+10} \qquad \underbrace{}_{\cancel{x^2}-16x+80}$$

$\Rightarrow \quad 14x = 70 \quad \Rightarrow \quad x = 5$.

$\Rightarrow \quad r = \sqrt{(5-1)^2+3^2} = \sqrt{4^2+3^2} = 5 \quad \Rightarrow \quad M = 0, \quad N = 10$.

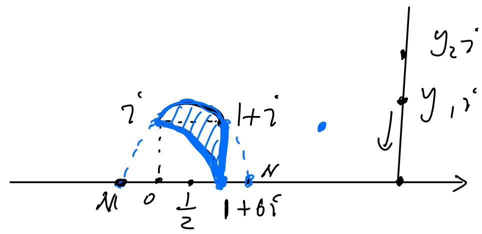
$$d(P,Q) = \left|\ln(P,Q;M,N)\right| = \left|\ln\frac{1}{6}\right| = \left|-\ln 6\right| = \ln 6$$

$$(P,Q;M,N) = \frac{P-M}{P-N} \Big/ \frac{Q-M}{Q-N} = \frac{(1+3i)-0}{(1+3i)-10} \cdot \frac{\overbrace{8+4i-10}^{-2+4i}}{8+4i}$$

$$= \frac{(1+3i)\cdot(-1+2i)}{(-9+3i)(4+2i)} = \frac{1}{3\cdot 2} \cdot \underbrace{\frac{1+3i}{-3+i}}_{-i} \cdot \underbrace{\frac{-1+2i}{2+i}}_{i}$$

$$= \frac{1}{6}\cdot(-i)\cdot i = \frac{1}{6}$$

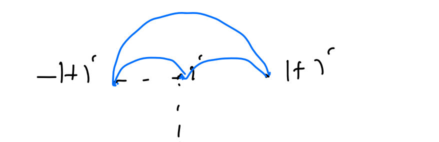
Ex: 7.47 $\qquad i\ ,\ 1+i\ ,\ 1$ $\qquad d(y_1 i, y_2 i) = \left|\ln \frac{y_2}{y_1}\right|$



$$r = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}$$

$$M = \frac{1}{2} - \frac{\sqrt{5}}{2}, \quad N = \frac{1}{2} + \frac{\sqrt{5}}{2} \qquad d(i, 1+i) = \left|\ln\left(i, 1+i; \frac{1}{2} - \frac{\sqrt{5}}{2}, \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\right|$$

Ex: 7.48.



$$\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$T_\gamma(z) = \frac{2z+1}{z+1}$$

$$T_\gamma(i) = \frac{2 \cdot i + 1}{1 \cdot i + 1} = \frac{(1+2i)(1-i)}{(1+i)(1-i)} = \frac{(1+2)+(2i-i)}{2} = \frac{3+i}{2}$$

$$T_\gamma(-1+i) = \frac{2 \cdot (-1+i)+1}{-1+i+1} = \frac{-1+2i}{i} = 2+i$$

$$\frac{-1+2i}{i} = \frac{2i}{i} + \frac{-1}{i}$$

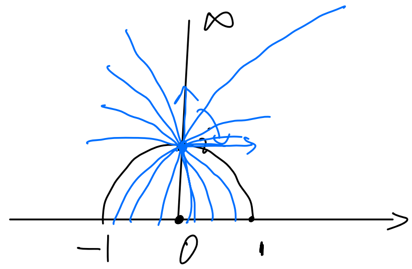
$(a+bi)(c+di) \qquad i^2 = -1$

$= (ac-bd)+(ad+bc)i$

$(a+bi)(a-bi) = a^2+b^2$

$$T_\gamma(1+i) = \frac{2 \cdot (1+i)+1}{1+i+1} = \frac{(3+2i)(2-i)}{(2+i)(2-i)} = \frac{(6+2)+(4-3)i}{2^2+1^2} = \frac{8+i}{5}$$

7.80: Isometry $i \to i$, $\infty \to 1$, $0 \mapsto -1$



$-\frac{1}{z}$: $\infty \to 0$, $i \mapsto i$, $0 \mapsto \infty$

$$(z, i; \infty, 0) = (w, i; 1, -1)$$

$$(z, i; \infty, 0) = \frac{z-\infty}{z-0} \Big/ \frac{i-\infty}{i-0} = \frac{i}{z}$$

$$(w, i; 1, -1) = \frac{w-1}{w+1} \Big/ \frac{i-1}{i+1} = \frac{1}{i} \cdot \frac{w-1}{w+1}, \quad \frac{i-1}{i+1} = i$$

$$\Rightarrow \quad -(w+1) = (w-1)z \quad \Rightarrow \quad wz + w = z - 1 \Rightarrow w = \frac{z-1}{z+1}$$

$$-(w+1) = -w-1, \quad (w-1)z = wz - z$$

$$i \mapsto \frac{i-1}{i+1} = i$$

$$\infty \to \frac{\infty-1}{\infty+1} = 1$$

$$0 \mapsto \frac{-1}{1} = -1.$$