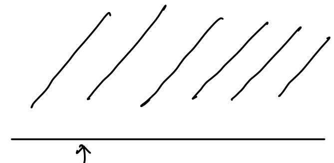
$\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$



$\mathbb{R} \cup \{\infty\} = \mathbb{RP}^1 = \partial_\infty \mathbb{H}$

$\operatorname{Isom}^+(\mathbb{H}) = \{ \text{orientation preserving isometries} \}$

$\cong \{ T_\gamma : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a, b, c, d \in \mathbb{R}, \ ad - bc > 0 \}$

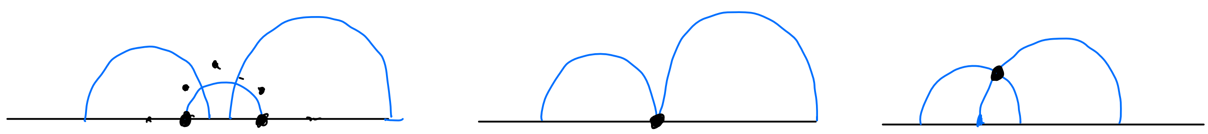
$T_\gamma(z) = \dfrac{az+b}{cz+d}$ $\qquad \boxed{T_\gamma(z) = z}$

- If T_γ has a fixed point in $\mathbb{H}$, then T_γ is a rotation

- If T_γ has no fixed point in $\mathbb{H}$, then T_γ is a translation.

  If T_γ has 2 fixed points on $\mathbb{RP}^1 = \partial_\infty \mathbb{H}$, then it is a hyperbolic translation

  If T_γ has 1 fixed point on $\mathbb{RP}^1 = \partial_\infty \mathbb{H}$, then it is a parabolic translation.



- Cross ratio : $\quad (z_1, z_2; z_3, z_4) = \dfrac{z_1 - z_3}{z_1 - z_4} \Big/ \dfrac{z_2 - z_3}{z_2 - z_4}$

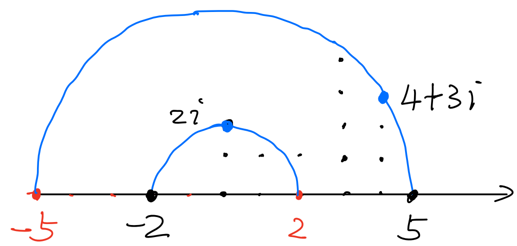
$$(T_\gamma(z_1), T_\gamma(z_2); T_\gamma(z_3), T_\gamma(z_4)) = (z_1, z_2; z_3, z_4)$$

Ex: Find the isometry T_γ of $\mathbb{H}$ which sends $2i$ to $3i+4$ and 2 to -5.

$2i \mapsto 3i+4$

$2 \mapsto -5$

$-2 \mapsto 5$



$z_1 \to w_1 = T_\gamma(z_1)$

$z_2 \to w_2 = T_\gamma(z_2)$

$z_3 \to w_3 = T_\gamma(z_3)$

$$(z, z_1; z_2, z_3) = (\underset{w}{T_\gamma(z)}, \underset{w_1}{T_\gamma(z_1)}; \underset{w_2}{T_\gamma(z_2)}, \underset{w_3}{T_\gamma(z_3)})$$

$$\frac{z-z_2}{z-z_3} \Big/ \frac{z_1-z_2}{z_1-z_3} = \frac{w-w_2}{w-w_3} \Big/ \frac{w_1-w_2}{w_1-w_3}$$

$\longrightarrow w = w(z)$

$$(z, \underline{2i}; 2, -2) = (w, \underline{3i+4}; -5, 5)$$

$$\frac{z-2}{z+2} \Big/ \underbrace{\frac{2i-2}{2i+2}}_{\frac{i-1}{i+1} = i} = \frac{w+5}{w-5} \Big/ \frac{3i+4+5}{3i+4-5} = \frac{9+3i}{-1+3i} = -3i$$

$$\boxed{\frac{1}{i} \cdot \frac{z-2}{z+2} = -\frac{1}{3i} \cdot \frac{w+5}{w-5}}$$

$$-3(z-2)(w-5) = (z+2)(w+5)$$

$$-3(zw - 5z - 2w + 10) = (zw + 5z + 2w + 10)$$

$$4zw + 2w - 6w = 15z - 30 - 5z - 10$$

$$w \cdot (4z-4) = 10z - 40$$

$$\boxed{w = \frac{10z-40}{4z-4} = \frac{5z-20}{2z-2}} \Leftarrow$$

$$2 \mapsto \frac{5\cdot 2-20}{2\cdot 2-2} = \frac{-10}{2} = -5$$

$$-2 \mapsto \frac{5\cdot(-2)-20}{2\cdot(-2)-2} = \frac{-30}{-6} = 5$$

$$\frac{5\cdot(2i)-20}{2\cdot 2i-2} = \frac{10i-20}{4i-2} = \frac{5i-10}{2i-1}$$

$$= \frac{(10-5i)(1+2i)}{(1-2i)(1+2i)} = \frac{20+15i}{5} = 4+3i \checkmark$$

$$T_\gamma(z) = \frac{5z-20}{2z-2} \qquad \gamma = \begin{pmatrix} 5 & -20 \\ 2 & -2 \end{pmatrix}$$

$$-3\,\frac{z-2}{z+2} = \frac{w+5}{w-5}$$

$$-3\,\frac{z-2}{z+2} = \frac{-3z+6}{z+2} = T_{\begin{pmatrix} -3 & 6 \\ 1 & 2 \end{pmatrix}}(z)$$

$$\frac{w+5}{w-5} = T_{\begin{pmatrix} 1 & 5 \\ 1 & -5 \end{pmatrix}}(w) = \begin{pmatrix} 1 & 5 \\ 1 & -5 \end{pmatrix}\cdot w$$

$$T_{\gamma_1} \circ T_{\gamma_2} = T_{\gamma_1 \gamma_2}$$

$$\Rightarrow w = \underbrace{\begin{pmatrix} 1 & 5 \\ 1 & -5 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -3 & 6 \\ 1 & 2 \end{pmatrix}} \cdot z$$

$$\begin{pmatrix} 1 & 5 \\ 1 & -5 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -3 & 6 \\ 1 & 2 \end{pmatrix} = \frac{1}{-10}\cdot\begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -3 & 6 \\ 1 & 2 \end{pmatrix} = \frac{1}{-10}\cdot\begin{pmatrix} 10 & -40 \\ 4 & -4 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} 5 & -20 \\ 2 & -2 \end{pmatrix}$$

$$w = T_{\begin{pmatrix} 5 & -20 \\ 2 & -2 \end{pmatrix}}(z) = \frac{5z-20}{2z-2} \quad \Leftarrow$$

Ex: T_γ that sends $2i \mapsto 3i$, $\infty \mapsto -1$,

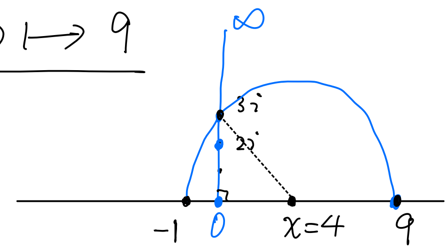
$0 \mapsto 9$

$$(z, 2i; \infty, 0) = (w, 3i; -1, 9)$$

$$\frac{z-\infty}{z-0} \Big/ \frac{2i-\infty}{2i} \qquad = \qquad \frac{w+1}{w-9} \Big/ \frac{3i+1}{3i-9}, \quad \frac{3i+1}{3i-9} = \frac{1}{3i}$$

$$\frac{z-\infty}{2i-\infty} \cdot \frac{2i}{z} \qquad\qquad 3i \cdot \frac{w+1}{w-9}$$

$$= \frac{2i}{z}$$



$$|x+1| = \sqrt{x^2+9}$$

$$x^2+2x+1 = x^2+9 \Rightarrow 2x = 8 \Rightarrow x = 4$$

$$\Rightarrow \frac{2\cancel{i}}{z} = 3\cancel{i} \cdot \frac{w+1}{w-9} \Rightarrow 2(w-9) = 3z(w+1)$$

$$\Rightarrow 3zw - 2w = -18 - 3z \Rightarrow \boxed{w = -3 \cdot \frac{z+6}{3z-2}}$$

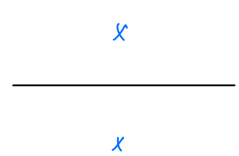
$$-3 \cdot \frac{2i+6}{3 \cdot 2i - 2} = -3 \cdot \underbrace{\frac{i+3}{3i-1}}_{= -i} = 3i \quad ✓ \qquad -3 \cdot \frac{0+6}{0-2} = -3 \cdot (-3) = 9. \quad ✓$$

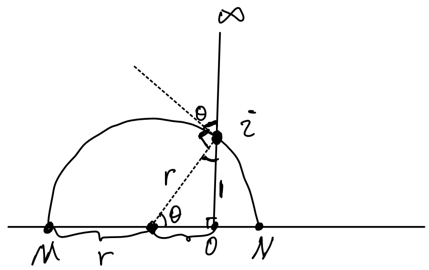
$$-3 \cdot \frac{\infty+6}{3\infty-2} = -3 \cdot \frac{1}{3} = -1 \quad ✓$$

$$w = T_\gamma(z) = -3 \cdot \frac{z+6}{3z-2} = z \Rightarrow -3z - 18 = 3z^2 - 2z$$

$$3z^2 + z + 18 = 0$$

$$\Rightarrow z = \frac{-1 \pm \sqrt{1^2 - 12 \cdot 18}}{6}$$





$$\boxed{\begin{aligned} i &\mapsto i \\ \infty &\mapsto M = -\cot\tfrac{\theta}{2} \\ 0 &\mapsto N = \tan\tfrac{\theta}{2} \end{aligned}}$$

$$r \cdot \sin\theta = 1 \Rightarrow r = \frac{1}{\sin\theta}$$

$$-r - r\cdot\cos\theta = -\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = -\frac{1+\cos\theta}{\sin\theta} = M$$

$$-\frac{1+\cos\theta}{\sin\theta} = -\frac{2\cdot\cos^2\frac{\theta}{2}}{2\cdot\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

$$r - r\cdot\cos\theta = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta} = N$$

$$\frac{1-\cos\theta}{\sin\theta} = \frac{2\cdot\sin^2\frac{\theta}{2}}{2\cdot\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\boxed{(z, i; \infty, 0) = \left(w, i; -\cot\tfrac{\theta}{2}, \tan\tfrac{\theta}{2}\right)}$$

$$(z, i; \infty, 0) = \frac{z-\infty}{z-0} \Big/ \frac{i-\infty}{i-0} = \frac{z-\infty}{i-\infty}\cdot\frac{i}{z} = \frac{i}{z}$$

$$\left(w, i; -\cot\tfrac{\theta}{2}, \tan\tfrac{\theta}{2}\right) = \frac{w+\cot\frac{\theta}{2}}{w-\tan\frac{\theta}{2}} \Bigg/ \frac{i+\cot\frac{\theta}{2}}{i-\tan\frac{\theta}{2}}$$

$$\cot\frac{\theta}{2} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}, \qquad \tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$\frac{i+\cot\frac{\theta}{2}}{i-\tan\frac{\theta}{2}} = \frac{\left(\tan\frac{\theta}{2}\cdot i+1\right)\cdot\frac{1}{\tan\frac{\theta}{2}}}{\left(i-\tan\frac{\theta}{2}\right)} = \frac{1}{i\cdot\tan\frac{\theta}{2}}$$

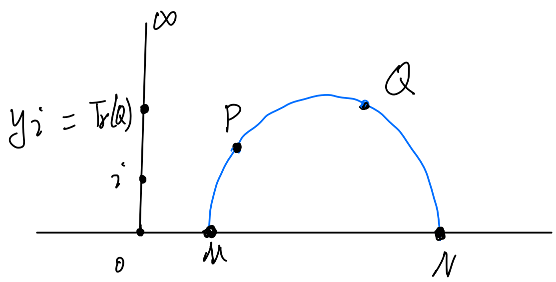
$$\rightsquigarrow \quad \boxed{w = \frac{\cos\frac{\theta}{2}\cdot z + \sin\frac{\theta}{2}}{-\sin\frac{\theta}{2}\cdot z + \cos\frac{\theta}{2}}}$$

$$\frac{\cancel{i}}{z} = \frac{w+\cot\frac{\theta}{2}}{w-\tan\frac{\theta}{2}}\cdot \cancel{i}\tan\frac{\theta}{2}$$

$$\gamma = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$\Rightarrow z\cdot w\cdot\tan\frac{\theta}{2} + z = w - \tan\frac{\theta}{2}$$

$$\Rightarrow z + \tan\frac{\theta}{2} = w\left(1 - z\cdot\tan\frac{\theta}{2}\right) \Rightarrow w = \frac{z+\tan\frac{\theta}{2}}{-z\cdot\tan\frac{\theta}{2}+1} = \frac{\cos\frac{\theta}{2}\cdot z + \sin\frac{\theta}{2}}{-\sin\frac{\theta}{2}\cdot z + \cos\frac{\theta}{2}} = T_\gamma(z)$$



$$d(P,Q) = d(\overset{\overset{T_\gamma(P)}{\shortparallel}}{i}, \overset{\overset{yi}{\shortparallel}}{T_\gamma(Q)})$$

$$= \ln \frac{y}{1} = \ln y$$

$$\boxed{\begin{aligned} d(P,Q) &= \left|\ln(Q,P;M,N)\right| \\ &= \left|\ln(P,Q;M,N)\right| \end{aligned}}$$

$$(yi, i; 0, \infty) = y$$

$$= \frac{yi-0}{yi-\infty} \Big/ \frac{i-0}{i-\infty} = \frac{yi}{i} \cdot \frac{i-\infty}{yi-\infty} = y$$

$$\underline{d(i, yi) = |\ln y| = |\ln(yi, i; 0, \infty)|}$$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

$$d(P,Q) = |\ln(Q,P;M,N)|$$

- $\underline{d(P,Q) = d(T_\gamma(P), T_\gamma(Q))}$
- $\underline{(P,Q;M,N) = (T_\gamma(P), T_\gamma(Q); T_\gamma(M), T_\gamma(N)).}$

$$\Rightarrow \quad d(P,Q) = |\ln(Q,P;M,N)|$$